Math 502 Homework 6

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Sorry it took me so long to post this. I do not know how hard this will turn out to be: I will mark it with overall class performance in mind. It will be due Apr. 22.

1. Prove that the union of a nested set of chains in a partial order $\leq$ is a chain. A chain is a set C such that for any $x, y \in C$ we have either $x \leq y$ or $y \leq x$; a nested collection of sets is a set A of sets which is a chain in the subset relation (for any $x, y \in A$, either $x \subseteq y$ or $y \subseteq x$).

2. Prove that for any ordinals α, β, γ if $\alpha + \beta = \alpha + \gamma$ then $\beta = \gamma$."

   You can probably prove this by transfinite induction, using the recursive definitions, but it can be proved using the set theoretic definition and structural properties of ordinals as well.

   Give a counterexample to "if $\beta + \alpha = \gamma + \alpha$ then $\beta = \gamma$.

3. Prove by transfinite induction: Every infinite ordinal can be expressed in the form $\lambda + n$, where λ is a limit ordinal and n is a finite ordinal, and moreover it can be expressed in this form in only one way (for this last part you might want to use the result of the previous problem).

4. Prove that the union of a countably infinite collection of countably infinite sets is countably infinite. Notice that you already know that $\mathbb{N} \times \mathbb{N}$ is a countable set.

   We give the result in more detail: suppose that F is a function with domain $\mathbb{N}$ and the property that each $F(n)$ is a countably infinite set. Show that $\bigcup\{F(n) \mid n \in \mathbb{N}\}$ is countable (that is, show that it is the range of a bijection with domain the set of natural numbers).

Hint: be very careful. It is fairly easy to see why this is true if you understand why $\mathbb{N} \times \mathbb{N}$ is a ocuntable set, but there is an application of the Axiom of Choice involved which you need to notice; in type theory or set theory without choice there may be countable collections of countable sets which have uncountable unions!

5. Use Zorn's Lemma to prove that every infinite set is the union of a pairwise disjoint collection of countably infinite sets.

   Then prove that if B is a collection of countably infinite sets, $|\bigcup B| = |\bigcup B| + |\bigcup B|$. (This exploits the fact that $|\mathbb{N}| = |\mathbb{N}| + |\mathbb{N}|$; it also requires the Axiom of Choice).

   Notice that this is another proof that $\kappa + \kappa = \kappa$ for any infinite cardinal κ. [It is also possible to prove $\kappa \cdot \kappa$ for κ infinite with essentially the same strategy.]