This is a take home exam. You should not seek help from any source other than the instructor, apart from the exception that you may seek advice from other students in the class about the computer component of the last question on the lab (question 1 here).

Please complete Homework 6, Homework 7, and the computer lab if you have not already done so. It is still worth your while to do these.

There are alternative questions here as indicated. If you do more than one of the alternatives, you will receive the grade for the best one.

In addition, you may drop one of the five questions or alternative pairs of questions (1,2,3,4/5,6/7) without penalty. If you do everything you get credit for your best work as usual.

The exam is due at the end of finals.

I will be holding office hours both in my office and in MG122 for extended periods of time this week and next week. You are welcome to come ask me questions about this exam as you have done with homework all term.

1. As already noted, the last exercise on the computer lab is a question on Test II as well.
2. Write a proof in Zermelo set theory with the modern form of the Axiom of Infinity and without Foundation that no natural number is an element of itself. This will of course be an induction proof using the definitions \(0 = \emptyset; n + 1 = n^+ = n \cup \{n\}\). Intense attention to “obvious” detail is needed at this level. Hint: it will be useful (and easy) to prove first (by induction of course) that all natural numbers are transitive.

Even more of a hint: the induction step looks like this. Suppose \(n \notin n\). Our goal is to show \(n + 1 = n \cup \{n\}\) is not an element of itself. Suppose otherwise for the sake of a contradiction. We suppose that is that \(n + 1 \in n + 1 = n \cup \{n\}\). So either 
\(n + 1 \in n + 1 = n \cup \{n\}\). So either \(n + 1 \in n\) (something bad happens...) or \(n + 1 = n\) (something bad happens...).
3. Formal syntax and semantics. Don’t be scared off: most of this question should be quite easy to work out from the definitions in the notes (possibly annoying of course). The last item in the third part is hard.

(a) Using the definitions of formal syntax in the notes, write out the mathematical object coding the formula

$$(\forall x^3_1.(\exists x^3_2.x^3_1 R^3_2 x^3_2)).$$

(b) What is the term or formula coded by

$$\langle 8, 1, \langle 0, 0, 1 \rangle, \langle 3, 2, \langle 0, 0, 1 \rangle, \langle 1, 0, 1 \rangle \rangle$$?

(c) Use the definitions of reference and satisfaction to evaluate the following expressions, if $D_0 = \{1, 2, 3\}$ and the following information about the environment and interpretation is given. Notice that we really do not need to worry about types in this example.

$A(0, 1) = 3$ (that is, the intended referent of $a^0_1$ is 3).

$P(0, 1) = \{1, 2\}$

$R(0, 1) = \{\langle 1, 1 \rangle, \langle 2, 2 \rangle, \langle 3, 3 \rangle\}$ (the equality relation).

$E(x^0_n) = 1$ for all $n$ (the environment $E$ assigns every type 0 variable the value 1).

Show the reasoning behind your evaluation in detail. The intended evaluations are quite obvious: the point is to show that the nasty definitions in the notes actually get us there, so detail must be seen. This is an exercise in step by step unpacking of definitions.

i. $R(E, a^0_1)$
ii. $R(E, x^0_1)$
iii. $V(E, P_1(a^0_1))$
iv. $V(E, P_1(x^0_1))$
v. $V(E, x^0_2 R_1 a^0_1)$
vi. $V(E, x^0_2 R_1 x^0_3)$
vii. $V(E, (\exists x^0_2.x^0_2 P_1 a^0_1))$
Answer one of the following two questions.

4. A classical argument that $|\mathcal{R}^2| = |\mathcal{R}|$ goes as follows. Suppose that it is granted that $|[0, 1]| = |\mathcal{R}|$ (this takes a wee bit of work, too, but not too much). So it suffices to prove that $|[0, 1]|^2 = |[0, 1]|$. Map the pair of numbers with decimal expansions $0.a_1a_2a_3\ldots$ and $0.b_1b_2b_3\ldots$ to the number with expansion $0.a_1b_1a_2b_2a_3b_3\ldots$. Unfortunately, this doesn’t quite give us the bijection we want due to problems with decimal expansions (explain). Give a corrected description of this map, taking into account bad features of decimal expansions, and explain why it is not a bijection from $[0, 1]^2$ to $[0, 1]$. Is it an injection? A surjection? Then use a theorem from the notes (giving all details of its application to this situation) to conclude that there is a bijection from $[0, 1]^2$ to $[0, 1]$.

5. In type theory, prove that for all ordinals $\alpha$ and $\beta$, if $\alpha + 1 = \beta + 1$ then $\alpha = \beta$. This is best proved by considering actual well-orderings and isomorphisms between them (not by transfinite induction).
Answer one of the following two questions (these depend on results to be discussed and proved in Thursday’s lecture)

6. Express the axioms of group theory in the language of first order logic (you do not need types and you do not need to use numerical codings). Groups are exactly models of this theory. A group is said to have torsion if there is an element \(g\) of the group and a natural number \(n\) such that \(g^n\) is the identity element \(e\) of the group. A group is said to be torsion-free if it does not have torsion. Prove that there is no formula \(\phi\) in our formal language for group theory which is true of exactly the groups with torsion. Hint: use compactness. Suppose that \(\phi\) is a formula which is true in every group with torsion. Consider the sentences \(\tau_n\) which say “there is a \(g\) such that \(g^n = e\)” for each concrete natural number \(n\). Notice (explain) that each of these sentence can be written in our formal language. Verify that the infinite set of sentences \(\{\phi, \neg \tau_1, \neg \tau_2, \neg \tau_3 \ldots\}\) satisfies the conditions of the Compactness Theorem (give details). Draw the appropriate conclusion.

Explain why this tells us that \((\exists n \in \mathbb{N}. g^n = e)\) is not equivalent to any sentence in our formal language for group theory.

7. The Löwenheim-Skolem Theorem tells us that every theory with a finite or countable language has a finite or countable model. Our untyped set theory has a countably infinite language, so has countably infinite models.

But in untyped set theory Cantor’s Theorem \(|A| < |\mathcal{P}(A)|\) holds. As an exercise in porting results from type theory to set theory, write out the proof of Cantor’s Theorem in untyped set theory. Hint: you do not need to make finicky use of the singleton operator in your argument.

Finally, if \(A\) is an infinite set in a model of untyped set theory, either \(A\) is not countably infinite (in which case we have an uncountable set) or \(A\) is countably infinite and \(|A| < |\mathcal{P}(A)|\), in which case \(\mathcal{P}(A)\) is an uncountable set (according to the model). Yet the whole model may be countably infinite, and so certainly any infinite subsets of the model are countably infinite. Why is this not a contradiction (this argument is called Skolem’s paradox). Hint: I’m using what look like the same words in different senses here; explain exactly how.