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Monotone weak Lindelöfness

Research Article

Maddalena Bonanzinga[1]*, Filippo Cammaroto[1]†, Bruno A. Pansera[1]‡

1 *Dipartimento di Matematica, Università di Messina, Messina, Italy*

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Abstract: The definition of monotone weak Lindelöfness is similar to monotone versions of other covering properties: X is monotonically weakly Lindelöf if there is an operator r that assigns to every open cover $\mathcal{U}$ a family of open sets $r(\mathcal{U})$ so that (1) $\bigcup r(\mathcal{U})$ is dense in X, (2) $r(\mathcal{U})$ refines $\mathcal{U}$, and (3) $r(\mathcal{U})$ refines $r(\mathcal{V})$ whenever $\mathcal{U}$ refines $\mathcal{V}$. Some examples and counterexamples of monotonically weakly Lindelöf spaces are given and some basic properties such as the behavior with respect to products and subspaces are discussed.

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1. Introduction

By a space we mean a topological space. Unless a weaker separation axiom is indicated specifically, we assume all spaces to be Tychonoff. By saying that the family of sets $\mathcal{A}$ refines a family of sets $\mathcal{B}$ (or that $\mathcal{B}$ is coarser than $\mathcal{A}$) we mean that every element of $\mathcal{A}$ is a subset of an element of $\mathcal{B}$; if $\mathcal{A}$ refines $\mathcal{B}$ we write $\mathcal{A} \prec \mathcal{B}$. If A is a subset of a space X and $\mathcal{B}$ is a family of subsets of X, we say that A refines $\mathcal{B}$ if A is a subset of some element of $\mathcal{B}$; in this case we write $A \prec \mathcal{B}$.

Recall that a space X is *weakly Lindelöf* [3] (or *wL*, for short) if for every open cover $\mathcal{U}$ of X there is a countable subfamily $\mathcal{U}_0 \subset \mathcal{U}$ with the union dense in X (i.e. $\overline{\bigcup \mathcal{U}_0} = X$).

Monotone versions of various covering properties have been previously considered in the literature. For example, monotone compactness was considered in [9, 10], monotone paracompactness and monotone countable paracompactness in [4–7, 16, 17].

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* *E-mail: milena_bonanzinga@hotmail.com*
† *E-mail: camfil@unime.it*
‡ *E-mail: bpansera@unime.it*

In particular, a space X is *monotonically Lindelöf* (or *mL*, for short) [1, 12, 15] if there is a function r, henceforth called an mL operator, that assigns to every open cover $\mathcal{U}$ of X a countable open cover $r(\mathcal{U})$ which refines $\mathcal{U}$ in such a way that $r(\mathcal{U})$ refines $r(\mathcal{V})$ whenever $\mathcal{U}$ refines $\mathcal{V}$.

In this paper we consider the monotone version of weak Lindelöfness, called mwL. We will see that in some aspects mwL is similar to mL, but in some others it behaves quite differently. This is not surprising because wL follows not only from Lindelöfness, but also from CCC.

2. mwL spaces – some examples and counterexamples

Definition 2.1.

A topological space X is *monotonically weakly Lindelöf* (or *mwL*, for short) if there is a function r, henceforth called an mwL operator, that assigns to every open cover $\mathcal{U}$ of X a countable family $r(\mathcal{U})$ which refines $\mathcal{U}$ in such a way that the union of $r(\mathcal{U})$ is dense in X and $r(\mathcal{U})$ refines $r(\mathcal{V})$ whenever $\mathcal{U}$ refines $\mathcal{V}$.

Throughout the paper, we will be using one technical lemma and two auxiliary notions that will help to show that certain spaces are *not* mwL.

Definition 2.2.

A subspace Y of a space X *is relatively mwL in* X if one can assign to every open cover $\mathcal{U}$ of Y by open sets of X a countable family $r(\mathcal{U})$ of open sets of X in such a way that $r(\mathcal{U})$ refines $\mathcal{U}$, $\overline{\bigcup r(\mathcal{U})} \supset Y$ and $r(\mathcal{U})$ refines $r(\mathcal{V})$ whenever $\mathcal{U}$ refines $\mathcal{V}$.

Definition 2.3.

Let X be a space and $p \in X$. X is *mwL at* p if there is an operator r_p that assigns to every nonempty family $\mathcal{U}$ of neighborhoods of p a countable family $r_p(\mathcal{U})$ of open sets such that $r_p(\mathcal{U})$ refines $\mathcal{U}$, $p \in \overline{\bigcup r_p(\mathcal{U})}$ and $r_p(\mathcal{G})$ refines $r_p(\mathcal{F})$ whenever $\mathcal{G}$ refines $\mathcal{F}$.

Theorem 2.4.

Let X be a regular and mwL space. Then there is an mwL operator r such that $r(\mathcal{U})$ consists of regular open sets for every $\mathcal{U}$.

Proof. Let r_0 be any mwL operator for X. We denote $\mathcal{U}' = \{V \subset X : V = \operatorname{Int}\overline{V}$ and there is $U \in \mathcal{U}$ such that $V \subset U\}$. Put $r(\mathcal{U}) = \{\operatorname{Int}\overline{W} : W \in r_0(\mathcal{U}')\}$. ☐

Of course mL implies mwL, therefore, in particular, the class of mwL spaces contains all second countable spaces, the one-point Lindelöfication of discrete space of cardinality ω_1, hereditary Lindelöf spaces with σ-(linearly-ordered by $\supset$)-base, separable GO-spaces (e.g. the Sorgefrey line).

Now we are going to show that the class of mwL spaces is much broader than the class of mL spaces: it contains many non-mL, sometimes even non-Lindelöf spaces.

Theorem 2.5.

Let X be a space and D be a countable dense subspace of X consisting of isolated points. Then X is mwL.

Proof. Let $\mathcal{U}$ be an open cover of X and D a countable dense subset of X. Then $r(\mathcal{U}) = \{\{d\} : d \in D\}$ is an mwL operator for X. ☐

Thus in particular we have

Example 2.6.
All Ψ-spaces are mwL.

In [12] it was proved that the one-point compactification of the discrete space of cardinality $\geq \omega_1$ is not mL. In contrast with this, we have the following

Theorem 2.7.
Let $\kappa \leq \mathfrak{c}$. Then the one-point compactification of the discrete space of cardinality κ is mwL.

Proof. Let $X = \kappa \cup \{p\}$ where the points of κ are isolated and every neighborhood of p has the form $\{p\} \cup (\kappa \setminus A)$ where $|A| < \omega$. Equip κ also with a second countable T_1 topology $\mathcal{T}$ (which is coarser than the discrete topology inherited by κ from X). Let $\mathcal{B}$ be a countable base of $\mathcal{T}$. For an open cover $\mathcal{U}$ of X, put $s(\mathcal{U}) = \{B \in \mathcal{B} : B \text{ refines } \mathcal{U}\}$ and $r(\mathcal{U}) = s(\mathcal{U}) \cup \left\{\{x\} : x \in \kappa \setminus \bigcup s(\mathcal{U})\right\}$. Then r is an mwL operator. (Indeed, it is easy to see that r is monotone, and $\bigcup r(\mathcal{U}) = \kappa$ for every $\mathcal{U}$.) ☐

Problem 2.8.
For what cardinals $\kappa > \mathfrak{c}$ is the one-point compactification of the discrete space of cardinality κ mwL?

Recall that $L(\kappa)$, *the one-point Lindelöfication of the discrete space of cardinality* κ is the set $X = \kappa \cup \{p\}$ equipped with the topology in which the points of κ are isolated and every neighborhood of p has the form $\{p\} \cup (\kappa \setminus A)$ where $|A| \leq \omega$.

Theorem 2.9 ([14]).
$L(\kappa)$ is mL iff $\kappa \leq \omega_1$.

Thus $L(\omega_1)$ is mwL.

Problem 2.10.
For what $\kappa > \omega_1$ is $L(\kappa)$ mwL?

Recall that any hereditarily Lindelöf space having a base σ-(linearly ordered by $\supset$) is mL; see [14]. Here is a similar result for monotone weak Lindelöfness:

Theorem 2.11.
Any space with σ-(linearly-ordered by $\supset$) π-base is mwL.

Proof. Without loss of generality assume that the π-base consists of nonempty sets. Since every linearly ordered by $\supset$ family of nonempty open sets contains a cofinal well ordered subfamily, let $\mathcal{B} = \bigcup\{\mathcal{B}_n : n \in \omega\}$ be a π-base of X consisting of nonempty sets such that for some ordinals κ_n (where $n \in \omega$), $\mathcal{B}_n = \{B_{n,\alpha} : \alpha < \kappa_n\}$ with $B_{n,\alpha} \supset B_{n,\beta}$ whenever $\alpha < \beta < \kappa_n$. For an open cover $\mathcal{U}$ of X and $n \in \omega$, put $\alpha_n(\mathcal{U}) = \min\{\alpha : B_{n,\alpha} \text{ refines } \mathcal{U}\}$. Then $r(\mathcal{U}) = \{B_{n,\alpha_n(\mathcal{U})} : n \in \omega\}$ defines an mwL operator for X. ☐

Corollary 2.12.
Every space X with a dense countable set D of points of countable character is mwL. Moreover, every space with a countable π-base is mwL.

Proof. For each point $x \in D$, fix $\mathcal{B}(x) = \{B_n(x) : n \in \omega\}$ a countable base at x, such that $B_{n+1}(x) \subset B_n(x)$. Then $\mathcal{B}(x)$ is well-ordered by $\supset$ and $\bigcup\{\mathcal{B}(x) : x \in D\}$ is a π-base as in Theorem 2.11. Then X is mwL. ☐

In particular, every separable first countable space is mwL.

In [13] there are consistent examples of countable spaces which are mL but not metrizable. However no such examples in ZFC are known. By Theorem 2.5, the space $\omega \cup \{p\}$ where $p \in \omega^*$ is a ZFC example of a countable space which is mwL but not metrizable.

Recall that the *Alexandroff duplicate* $AD(X)$ of the topological space X is the set $X \times 2$ where the points of $X \times \{1\}$ are isolated while a basic neighborhood of a point $(x, 0) \in X \times \{0\}$ takes the form $(U \times 2) \setminus \{(x, 1)\}$ where U is a neighborhood of x in X.

Theorem 2.13.
If X is a second countable space, then $AD(X)$ is mwL.

Proof. Let $\mathcal{B}$ be a countable base of X. Let $\mathcal{U}$ be an open cover of $AD(X)$. Put $s(\mathcal{U}) = \{B \times 2 : B \in \mathcal{B}$ and $B \times 2$ refines $\mathcal{U}\}$. $AD(X) \setminus \bigcup\mathcal{U}$ is at most countable. Put $r(\mathcal{U}) = s(\mathcal{U}) \cup \left\{\{(p, 1)\} : (p, 1) \notin \bigcup s(\mathcal{U})\right\}$. r is an mwL operator for $AD(X)$. ☐

3. Subspaces

3.1. Closed and regular closed subspaces

Recall that mL is preserved by closed subspaces. That mwL is not preserved by closed subspaces can be seen already from Example 2.6 (because a Ψ-space contains an uncountable closed discrete subspace which of course is not mwL). But we can prove more.

Theorem 3.1.
Every Tychonoff space of weight $\leq \mathfrak{c}$ can be embedded in an mwL Tychonoff space as a closed subspace.

Proof. Let Y be a space with $w(Y) \leq \mathfrak{c}$. Without loss of generality we assume that $Y \subset I^{\mathfrak{c}}$. Let C be a dense countable subspace of $I^{\mathfrak{c}}$. Put $X = (Y \times \{0\}) \cup (C \times \{1\}) \subset AD(I^{\mathfrak{c}})$. Then X has an open dense countable subspace $C \times \{1\}$ consisting of isolated points. So X is mwL by Proposition 2.5. Now $Y \times \{0\}$ is a closed nowhere dense subspace of X homeomorphic to Y. ☐

So we see that an mwL space may have cardinality $\geq 2^{\mathfrak{c}}$, extent $\geq \mathfrak{c}$ and character $\geq \mathfrak{c}$. In [14] Levy and Matveev asked if the weight of every mL space is not greater than $\mathfrak{c}$ and noted that a similar question makes sense apparently for any other "reasonable" cardinal function (except for those for which the answer is trivially affirmative, e.g. the extent). So now we see that for mwL the situation may be different, but how much different is it?

Problem 3.2.
If X is an mwL space, does it follow that $w(X) \leq \mathfrak{c}$? What can one say about other cardinal invariants of X?

For regular closed sets, the situation is different to that of closed sets in general.

Theorem 3.3.
Let X be an mwL space and Y a regular closed subset of X. Then Y is mwL.

Proof. Let $\mathcal{V}$ be an open cover of Y. Put $s(\mathcal{V}) = \{V \cup (X \setminus Y) : V \in \mathcal{V}\}$. Then $s(\mathcal{V})$ is an open cover of X, and passing from $\mathcal{V}$ to $s(\mathcal{V})$ is a monotone operation. Let r_X be an mwL operator for X. Then $r_Y(\mathcal{V}) = \{U \cap Y : U \in r_X(s(\mathcal{V}))\}$ shows that Y is mwL. (That r_Y is monotone and $r_Y(\mathcal{V})$ refines $\mathcal{V}$ is obvious. It remains to note that $r_Y(\mathcal{V})$ is dense in Y because $\operatorname{Int}_X Y$ is dense in Y). $\square$

3.2. Dense subspaces

Theorem 3.4.
Let X be a space and Y be an open dense mwL subspace of X. Then X is mwL.

Proof. Since Y is mwL, there is an mwL operator r_Y for Y. Let $\mathcal{U}$ be an open cover of X. Put $s(\mathcal{U}) = \{U \cap Y : U \in \mathcal{U}\}$ and $r(\mathcal{U}) = r_Y(s(\mathcal{U}))$. Then r is an mwL operator for X. $\square$

Next we consider the implications of removing the openness condition in the previous result. Removing this condition produces different results for regular spaces and for Hausdorff spaces.

Theorem 3.5.
Let X be a T_3 space and Y be a dense mwL subspace of X. Then X is mwL.

Proof. Let $\mathcal{U}$ be an open cover of X. Put $\widetilde{\mathcal{U}} = \{V : V \text{ is regular open set in } X \text{ and there is } U \in \mathcal{U} \text{ with } V \subset U\}$. Then $\widetilde{\mathcal{U}}$ is a cover of X by regular open sets, $\widetilde{\mathcal{U}}$ refines $\mathcal{U}$, and the operation $\mathcal{U} \to \widetilde{\mathcal{U}}$ is monotone. Put $\widetilde{\mathcal{U}}_Y = \left\{V \cap Y : V \in \widetilde{\mathcal{U}}\right\}$; this is a cover of Y by regular open sets of Y. Let r_Y be an mwL operator for Y. Put $r(\mathcal{U}) = \left\{\operatorname{Int}_X(\operatorname{cl}_X W) : W \in r_Y\left(\widetilde{\mathcal{U}}_Y\right)\right\}$. Then r is an mwL operator for X. $\square$

Example 3.6.
There is a Hausdorff space X and a dense subspace $Y \subset X$ such that Y is mwL but X is not.

Proof. Let $\mathbb{R} = \mathbb{Q} \cup \mathbb{P}$ be the usual real line, where $\mathbb{Q}$ is the set of the rational numbers and $\mathbb{P}$ the set of the irrational numbers. Let $\mathcal{E}$ be the usual topology on $\mathbb{R}$. We define a finer topology $\mathcal{T} \supset \mathcal{E}$ as follows. Let $\mathbb{P} = \bigsqcup_{\alpha<\mathfrak{c}} P_\alpha$ be a partition of $\mathbb{P}$ into $\mathfrak{c}$ many dense (with respect to $\mathcal{E}$) subspaces. The topology $\mathcal{T}$ is generated by the base $\mathcal{B} = \left\{U \setminus \bigcup_{\alpha\in F} P_\alpha : U \in \mathcal{E},\ F \text{ is a finite subset of } \mathfrak{c}\right\}$. The new topology is constructed so that every nonempty open set intersects all but finitely many P_αs.

Since $\mathcal{T}|_{\mathbb{Q}} = \mathcal{E}|_{\mathbb{Q}}$, it follows that $\mathbb{Q}$ is second countable and hence mwL. Moreover, every nonempty element of $\mathcal{B}$ intersects $\mathbb{Q}$, so $\mathbb{Q}$ is dense in $(\mathbb{R}, \mathcal{T})$.

Since $\mathcal{T} \supset \mathcal{E}$, $(\mathbb{R}, \mathcal{T})$ is Hausdorff. It remains to show that $(\mathbb{R}, \mathcal{T})$ is not mwL. Suppose the contrary; let r be an mwL operator. The family $\mathcal{U}_0 = \{\mathbb{R} \setminus P_\alpha : \alpha \in \mathfrak{c}\}$ is an open cover of $(\mathbb{R}, \mathcal{T})$. For every $U \in r(\mathcal{U}_0)$, pick $\alpha_U \in \mathfrak{c}$ so that $U \subset X \setminus P_{\alpha_U}$ (this is possible because $r(\mathcal{U}_0)$ refines $\mathcal{U}_0$). Put $C_0 = \{\alpha_U : U \in r(\mathcal{U}_0)\}$. Then C_0 is a countable subset of $\mathfrak{c}$.

Next, $\mathcal{U}_1 = \{\mathbb{R} \setminus P_\alpha : \alpha \in \mathfrak{c} \setminus C_0\}$ is an open cover of $(\mathbb{R}, \mathcal{T})$. For every $U \in r(\mathcal{U}_1)$, pick $\alpha_U \in \mathfrak{c} \setminus C_0$ so that $U \subset X \setminus P_{\alpha_U}$. Put $C_1 = \{\alpha_U : U \in r(\mathcal{U}_1)\}$. Then C_1 is a countable subset of $\mathfrak{c}$ disjoint from C_0. Etc.

Similarly, we define for all $n \in \omega$ covers $\mathcal{U}_n = \left\{\mathbb{R} \setminus P_\alpha : \alpha \in \mathfrak{c} \setminus \bigcup_{m<n} C_m\right\}$ and pairwise disjoint countable subsets $C_n \subset \mathfrak{c}$ so that for every $U \in r(\mathcal{U}_n)$ there is $\alpha_U \in C_n$ such that $U \cap P_{\alpha_U} = \emptyset$.

Finally, put $\mathcal{U}_\omega = \left\{\mathbb{R} \setminus P_\alpha : \alpha \in \mathfrak{c} \setminus \bigcup_{m<\omega} C_m\right\}$. Let $U \in r(\mathcal{U}_\omega)$, $U \neq \emptyset$. Since for every $n \in \omega$, $\mathcal{U}_\omega$ refines $\mathcal{U}_n$, and for every $n \in \omega$, $r(\mathcal{U}_\omega)$ refines $r(\mathcal{U}_n)$, and thus for every n there is $U_n \in r(\mathcal{U}_n)$ such that $U \subset U_n \subset X \setminus P_{\alpha_{U_n}}$. So U is a nonempty $\mathcal{T}$-open set that does not intersect infinitely many sets P_α. This contradicts the definition of the topology $\mathcal{T}$. $\square$

3.3. Countable subspaces

It was shown in [14] that a countable subspace of an mL space is mL. In contrast to this, we have

Example 3.7.
A countable subspace of an mwL space need not be mwL.

Indeed, as we will see below, a dense countable subspace, call it C, of $2^{\mathfrak{c}}$ is not mwL. Then the Alexandroff duplicate $AD(C)$ is mwL since it contains a dense countable subspace $C\times\{1\}$ consisting of isolated points; thus $AD(C)$ is mwL. On the other hand, $AD(C)$ contains a subspace $C\times\{0\}$ homeomorphic to C.

The next theorem improves Corollary 2.12.

Theorem 3.8.
If X has a dense countable subspace D such that X is mwL at the points of D, then X is mwL.

Proof. For every $d \in D$, let r_d be an mwL operator at d. Let $\mathcal{U}$ be an open cover of X. For every $d \in D$, put $\mathcal{U}_d = \{U : U \in \mathcal{U},\ d \in U\}$. Then, $r(\mathcal{U}) = \bigcup_{d\in D} r_d(\mathcal{U}_d)$ defines an mwL operator for X. ☐

Recall that a sequence of sets $\{A_n : n \in \omega\}$ converges to a point p if each neighborhood of p contains all but finitely many sets A_n.

Corollary 3.9.
If X has a dense countable subspace D such that for every $d \in D$ there exists a sequence of open sets of X converging to d, then X is mwL.

Proof. Pick $d \in D$. Let $\{A_n : n \in \omega\}$ be a sequence of open sets converging to d. Given a nonempty family $\mathcal{U}$ of neighborhoods of d, put $r_d(\mathcal{U}) = \{A_n : A_n \subset U \text{ for some } U \in \mathcal{U}\}$. r_d is an mwL operator for X at d. ☐

Note that the previous corollary also follows from the countable π-base part of Corollary 2.12.

3.4. Unions of subspaces

For completeness of our exposition, we provide a proof of the following widely accepted lemma.

Lemma 3.10.
Let $X = X_1 \cup X_2$ be any space. Then there exist open sets O_1 and O_2 such that $X_i \cap O_i$ is dense in O_i for $i = 1, 2$, and $\overline{O_1} \cup \overline{O_2} = X$.

Proof. Let $\mathcal{O}_i = \{O : O \text{ is an open set in } X \text{ such that } X_i \text{ is dense in } O\}$, $i = 1, 2$. We claim that $\bigcup\mathcal{O}_1\cup\bigcup\mathcal{O}_2$ is dense in X. By contradiction, assume that $\bigcup\mathcal{O}_1 \cup \bigcup\mathcal{O}_2$ is not dense in X, i.e. $\overline{\bigcup\mathcal{O}_1 \cup \bigcup\mathcal{O}_2} \neq X$. Then $U \subset X \setminus \overline{\bigcup\mathcal{O}_1 \cup \bigcup\mathcal{O}_2}$ is a nonempty open set in X. Then $U \notin \mathcal{O}_i$, $i = 1, 2$. Then $X_1 \cap U$ is not dense in U. Then $U' = U \setminus \overline{X_1} \subset X_2$ is a nonempty open set such that $X_2 \cap U'$ is dense in U'. Then $U' \in \mathcal{O}_2$, a contradiction. ☐

Theorem 3.11.
If a regular space X is a finite union of mwL spaces X_i, $i = 1, \dots, k$, then X is mwL.

Proof. It suffices to consider the case $k=2$. By Lemma 3.10 there exist O_1 and O_2, open sets in X, such that $X_i \cap O_i$ is dense in O_i, $i=1,2$, and $\overline{O_1} \cup \overline{O_2} = X$. Then $\overline{O_i} \cap X_i$ is a regular closed set in X_i, $i=1,2$. Then by Proposition 3.3, $\overline{O_i} \cap X_i$ is mwL and by Proposition 3.5, $\overline{O_i}$ is mwL, $i=1,2$. Let r_i be an mwL operator for $\overline{O_i}$, $i=1,2$. Let $\mathcal{U}$ be an open cover of X and $\mathcal{U}_i = \{U \cap \overline{O_i} : U \in \mathcal{U}\}$ be an open cover of $\overline{O_i}$, $i=1,2$. Then $r(\mathcal{U}) = \bigcup_{i=1}^{2}\{V \cap O_i : V \in r_i(\mathcal{U}_i)\}$ is an mwL operator for X. ☐

Example 3.12.

A countable union of mwL spaces does not have to be mwL.

As we already mentioned, a countable dense subset X of $2^{\mathfrak{c}}$ it is not mwL. So $X = \bigcup_{n\in\omega}\{x_n\}$, $\{x_n\}$ are mwL spaces.

Remark.

It is interesting to compare Theorem 3.11 with the fact that the union of two mL subspaces need not be mL: it is enough to note that there exist countable spaces with single non-isolated point which are not mL [14]. The same example shows that the union of countably many closed mL subspaces need not be mL even if the space is stratifiable [8]. The question whether the union of two *closed* mL subspaces must be mL remains open [14].

4. Products and dense subspaces of products

4.1. Finite and countable products

The question when a product of two mwL spaces is mwL has yet to be studied systematically. Here we present only one simple result (a result which is interesting because the product of an mL space and a convergent sequence does not have to be mL [12]).

Theorem 4.1.

The product of an mwL space X and a space Y having a dense countable set of isolated points is mwL.

Proof. Let $D = \{d_n : n \in \omega\}$ be a dense countable set of isolated points of Y. We observe that each subspace $X \times \{d_n\}$, $n \in \omega$, is homeomorphic to X, clopen in $Z = X \times D$ and mwL. Then Z is the discrete sum of countably many mwL spaces, and thus Z is mwL. Further, Z is an open, dense and mwL subspace of $X \times Y$. Then $X \times Y$ is mwL. ☐

Now we consider several specific facts about products. The next theorem may have applications not only to the study of products.

Theorem 4.2.

If $(X,\mathcal{T})$ is such a space that there is a weaker second countable topology $\mathcal{E}$ on X with the property that, for every $U \in \mathcal{T}$, $U \subset \mathrm{cl}_{\mathcal{T}}(\mathrm{Int}_{\mathcal{E}} U)$, then $(X,\mathcal{T})$ is mwL.

Proof. Let $\mathcal{U}$ be an open cover of X with respect to the topology $\mathcal{T}$ and $\mathcal{B}$ be a countable base of X with respect to the topology $\mathcal{E}$. The family $r(\mathcal{U}) = \{O \in \mathcal{B} : O \subset U \text{ for some } U \in \mathcal{U}\}$ shows that $(X,\mathcal{T})$ is an mwL space. Indeed, obviously $r(\mathcal{U}) \prec \mathcal{U}$. We have to prove that $\mathrm{cl}_{\mathcal{T}}\left(\bigcup r(\mathcal{U})\right) = X$. Let $x \in X$ and $V \in \mathcal{T}$ such that $x \in V$. Fix $U \in \mathcal{U}$ such that $x \in U$ and put $W = V \cap U$. Since $W \subset \mathrm{cl}_{\mathcal{T}}(\mathrm{Int}_{\mathcal{E}} W)$, there exists $O \in \mathcal{B}$ such that $O \subset W$. Hence $O \subset U$ and then $O \in r(\mathcal{U})$; also $O \cap V \neq \emptyset$. It is obvious that if $\mathcal{V} \prec \mathcal{U}$ then $r(\mathcal{V}) \prec r(\mathcal{U})$. ☐

Corollary 4.3.

Every finite power or countable power of the Sorgenfrey line is mwL.

Proof. Because the Euclidean topology and the Sorgenfrey topology on finite or countable power of the line are related as in Proposition 4.2. ☐

It is worth noting that Ψ^ω (where Ψ is a Ψ-space), $(\beta\omega)^\omega$ etc. are mwL. This follows from

Theorem 4.4.

Let X be a regular space having a second countable dense subspace. Then X^ω is mwL. (In particular, for every regular space X having a dense countable subspace consisting of isolated points, X^ω is mwL.)

Proof. Let D be a second countable dense subspace of a regular space X. Then D^ω is a second countable (hence mwL) dense subspace of X^ω. Then by Theorem 3.5, X^ω is mwL. ☐

4.2. Uncountable products

It was shown in [14] that if X is a dense subspace in the product of uncountably many nontrivial (that is T_1 and consisting of at least two points) factors, then X is not mL at any point. Here we verify that the same is true for mwL; the argument is similar to that from [14].

Theorem 4.5.

If X is a dense subspace in the product $Y = \prod_{\alpha\in A} Y_\alpha$, where $|A| > \omega$ and for each α, Y_α is a regular space and $|Y_\alpha| \geq 2$, then X is not mwL at any point.

Proof. For a subset $U \subset Y$, put $A(U) = \{\alpha \in A : \pi_\alpha(U) \neq Y_\alpha\}$. It follows from the definition of the Tychonoff product topology that for every nonempty open set U, the set $A(U)$ is finite. For a subset $V \subset X$, and $\alpha \in A$, we say that V depends on $\alpha \in A$ if $\pi_\alpha(V)$ is not dense in Y_α. Put $A'(V) = \{\alpha \in A : V \text{ depends on } \alpha\}$. Since X is dense in Y, it follows that

$$\text{for every nonempty open set } V \text{ in } X, \text{ the set } A'(V) \text{ is finite.} \tag{$\star$}$$

Let $p \in X$. Suppose there exists an mwL operator r_p at p.

For every $\alpha \in A$, we pick two distinct points $x_0^\alpha, x_1^\alpha \in Y_\alpha$ and open sets V_0^α, W_0^α so that $x_0^\alpha \in W_0^\alpha \subset \overline{W_0^\alpha} \subset V_0^\alpha \subset \overline{V_0^\alpha} \subset X \setminus \{x_1^\alpha\}$. Put $V_1^\alpha = X \setminus \overline{W_0^\alpha}$. Then the sets V_0^α and V_1^α are nonempty and none of them is dense in Y_α. Put $v_\alpha = \{V_i^\alpha : i = 0, 1\}$ and $\mathcal{V}_\alpha = \{\pi_\alpha^{-1}(V) : V \in v_\alpha\}$. Then v_α is an open cover of Y_α and $\mathcal{V}_\alpha$ is an open cover of Y. By induction on $n \leq \omega$, we construct the following:

- pairwise disjoint nonempty countable subsets $C_n \subset A$,
- open covers $\widetilde{\mathcal{U}}_n$ of Y,
- open covers $\mathcal{U}_n$ of X,

so that the following conditions will be satisfied:

1. $\widetilde{\mathcal{U}}_n = \bigcup\{\mathcal{V}_\alpha : \alpha \in A \setminus \bigcup_{m<n} C_m\}$;
2. $\mathcal{U}_n = \widetilde{\mathcal{U}}_n|_X$ (this means that $\mathcal{U}_n = \{U \cap X : U \in \widetilde{\mathcal{U}}_n\}$);
3. for every $U \in r_p(\mathcal{U}_n)$, there is $\alpha_U \in C_n$ such that U depends on α_U.

As the base of induction, put $\widetilde{\mathcal{U}}_0 = \bigcup\{\mathcal{V}_\alpha : \alpha \in A\}$ and define $\mathcal{U}_0$ according to condition 2. For every $U \in r_p(\mathcal{U}_0)$, pick $\alpha_U \in A$ so that U depends on α_U (such α_U exists because $U \in r_p(\mathcal{U}_0) \prec \mathcal{U}_0 \prec \widetilde{\mathcal{U}}_0$ and thus $U \subset \pi_\alpha^{-1}(V_i^\alpha)$ for some α and i). Put $C_0 = \{\alpha_U : U \in r_p(\mathcal{U}_0)\}$. Since $r_p(\mathcal{U}_0)$ is countable, so is C_0.

Now suppose $n \leq \omega$ and $C_m, \widetilde{\mathcal{U}}_m$ and $\mathcal{U}_m$ have been defined for all $m < n$. Put $B_n = A \setminus \bigcup_{m<n} C_m$ and define $\widetilde{\mathcal{U}}_n$ and $\mathcal{U}_n$ according to conditions 1. and 2. Since A is uncountable and all C_m are countable, B_n is nonempty and thus $\widetilde{\mathcal{U}}_n$ covers

Y and $\mathcal{U}_n$ covers X. For every $U \in r_p(\mathcal{U}_n)$, pick $\alpha_U \in B_n$ so that U depends on α_U. Put $C_n = \{\alpha_U : U \in r_p(\mathcal{U}_n)\}$. Then C_n is nonempty, countable, and disjoint from each of previous C_m. This concludes the construction.

Note that whenever $n > m$, $\widetilde{\mathcal{U}}_n \prec \widetilde{\mathcal{U}}_m$, therefore $\mathcal{U}_n \prec \mathcal{U}_m$ and in turn $r_p(\mathcal{U}_n) \prec r_p(\mathcal{U}_m)$. Let $U \in r_p(\mathcal{U}_\omega)$. Then for every $n < \omega$, there is $U_n \in r_p(\mathcal{U}_n)$ such that $U \subset U_n$. Since U_n depends on $\alpha_{U_n} \in C_n$, U depends on α_{U_n} for every n. Since the sets C_n are pairwise disjoint, it follows that U depends on infinitely many coordinates, that is $A'(U)$ is infinite; a contradiction to ($\star$). ☐

The following are immediate corollaries from Theorem 4.5:

- 2^κ is not mwL whenever $\kappa > \omega$.
- A dense countable subspace in 2^κ (where $\omega_1 \leq \kappa \leq \mathfrak{c}$) is an example of a countable space which is not mwL (at any point).

Also, since $C_p(X)$ is dense in $\mathbb{R}^X$, by Theorem 4.5 we have:

- $C_p(X)$ is mwL iff X is countable.

Remark 4.6.

One of the referees pointed out a similarity between the proof of Example 3.6 and Theorem 4.5 and asked if some common lemma could be extracted. Actually, the proof of Example 3.6 and Theorem 4.5 is similar also to some arguments in [11, 12, 14]. Perhaps a common property of partially ordered sets that applies to all these arguments can be found. This might be a subject for further research.

5. Mappings

Recall that a condensation is a continuous bijection.

Example 5.1.

An mwL space may condense onto a non-mwL space.

Proof. Let D be a dense countable subspace of the space $2^\mathfrak{c}$. Consider the following subspace Y of $AD(2^\mathfrak{c})$: $Y = \big((2^\mathfrak{c} \setminus D)\times\{0\}\big) \cup (D\times\{1\})$ and the projection $\pi : AD(2^\mathfrak{c}) \to 2^\mathfrak{c}$. Then $\pi|_Y$ is a condensation from an mwL space to a non-mwL space. ☐

Example 5.2.

mwL is not preserved by perfect finite-to one maps.

Let D be a dense countable subspace of the space $2^\mathfrak{c}$. Consider the following subspace Z of $AD(2^\mathfrak{c})$: $Z = \big(2^\mathfrak{c}\times\{0\}\big) \cup (D\times\{1\})$ and the projection $\pi\colon AD(2^\mathfrak{c}) \to 2^\mathfrak{c}$. Then $\pi|_Z$ is a perfect and ≤ 2 to 1 map from a compact space to a non-mwL space.

Theorem 5.3.

A continuous open image of an mwL space is mwL.

Proof. Let f be a continuous open mapping from an mwL space X onto a space Y, r an mwL operator on X and $\mathcal{V}$ an open cover of Y. Then $\mathcal{U} = \big\{f^{-1}(V) : V \in \mathcal{V}\big\}$ is an open cover of X. Put $r_Y(\mathcal{V}) = \{f(U) : U \in r(\mathcal{U})\}$. Then r_Y is an mwL operator for Y. ☐

Problem 5.4.
Is mwL preserved by closed irreducible maps?

Problem 5.5.
Suppose all continuous images of X are mwL. What can be concluded about X?

6. Some more questions

Problem 6.1.
Is every mwL LOTS or GO-space mL? In particular, is $[0, \omega_1]$ mwL?

Problem 6.2.
Is $\omega^\star = \beta\omega \setminus \omega$ mwL?

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