Generating Sets of Cofinitary Groups

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Outline

- Definitions and Basics.
- Two Questions, their meaning and motivation.
- Some ideas from the proofs.
Definition
Sym(\(\mathbb{N}\)): the group of bijections \(\mathbb{N} \rightarrow \mathbb{N}\) (permutations) with operation composition.

\(f \in \text{Sym}(\mathbb{N})\) is cofinitary iff \(f\) is the identity or has only finitely many fixed points.

\(G \leq \text{Sym}(\mathbb{N})\) is a cofinitary group (sharp group) iff all \(g \in G\) are cofinitary.

\(G \leq \text{Sym}(\mathbb{N})\) is a maximal cofinitary group (MCG) iff \(G\) is a cofinitary group and is not properly contained in another cofinitary group.
(Adeleke, Truss) A maximal cofinitary group cannot be countable.

(P. Neumann) There is a cofinitary group of size $|\mathbb{R}|$.

Any cofinitary group is contained in a maximal cofinitary group.

(Yi Zhang) If $|\mathbb{N}| < \kappa \leq |\mathbb{R}|$ then it is consistent that there is an MCG $G$ with $|G| = \kappa$.

A cofinitary group is an almost disjoint (eventually different) family of permutations that is also a group.
If a cofinitary group has infinitely many orbits, then it is not maximal.

Martin’s Axiom: any finite number of finite and infinite orbits are possible.

Martin’s Axiom: there exists a locally finite maximal cofinitary group.
Two Questions (Vershik)

Does there exist a dense maximal cofinitary group?

Does there exist a noncomputable cofinitary group with a computable set of generators?
Does there exist a dense maximal cofinitary group?

Topology generated by basic open sets

\[ [s] = \{ f \in \text{Sym}(\mathbb{N}) : s \subseteq f \}. \]

Hierarchies above this.
Does there exist a noncomputable cofinitary group with a computable set of generators?

**Theorem (Su Gao, Yi Zhang)**

\( V = L \) implies there is an mcg with a coanalytic generating set.

**Theorem (K)**

\( V = L \) implies there is a coanalytic mcg.
Does there exist a noncomputable cofinitary group with a computable set of generators?

Computable set of generators:
There is a computable function \( \varphi \) with two inputs such that \( \{ n \mapsto \varphi(m, n) : m \in \mathbb{N} \} \) is this set of functions.

Computable group \( \langle G, \circ \rangle \):
There is a computable function \( \psi \) with two inputs such that \( (n, m) \mapsto \psi(n, m) \) gives a group structure on \( \mathbb{N} \), and this group is isomorphic to \( \langle G, \circ \rangle \).
Constructing CG

\[ G_{\alpha+1} = \langle G_\alpha, g_\alpha \rangle = (G_\alpha \ast F(x))[x := g_\alpha] \]

\[ g_\alpha = \bigcup_{s \in \mathbb{N}} g_{\alpha,s} \]

Study

\[ w(g_\alpha,s) \rightsquigarrow w(g_\alpha,s+1) \]

*good extension*: no unavoidable new fixed points.

\[ w = u^{-1}vu \]
$G$ a countable cofinitary group, $f \in \text{Sym}(\mathbb{N}) \setminus G$ such that $\langle G, f \rangle$ is cofinitary, $p : \mathbb{N} \rightarrow \mathbb{N}$ finite, and $w \in G \ast F(x)$.

**Lemma (Domain Extension)**

*For all $n \not\in \text{dom}(p)$ for all but finitely many $k$, $p \cup \{(n, k)\}$ is a good extension of $p$ w.r.t. $w$.***

**Lemma (Range Extension)**

*For all $k \not\in \text{ran}(p)$ for all but finitely many $n$, $p \cup \{(n, k)\}$ is a good extension of $p$ w.r.t. $w$.***

**Lemma (Hitting $f$)**

*For all but finitely many $n$, $p \cup \{(n, f(n))\}$ is a good extension of $p$ w.r.t. $w$.***
Theorem

MA implies there exists an mcg into which every countable group embeds.

\[ G = \star_{\alpha < c} G_\alpha. \]

with \( G_\alpha \) of all different isomorphism types.
Does there exist a dense maximal cofinitary group?

Let $\langle s_n : n \in \mathbb{N} \rangle$ be an enumeration all all finite injections $\mathbb{N} \rightarrow \mathbb{N}$. Start the different generators with these.
Construction CG with relations

\[ \bar{R} = \langle R_s \subseteq F(\bar{x}) : s \in \mathbb{N} \rangle \]
\[ G_s = F(\bar{x})/R_s \]
\[ W_{s,\text{Id}} = \{ w \in F(\bar{x}) : w =_{G_s} \text{Id} \} \]

\( \bar{R} \) is a demure sequence iff

- for all \( t \in \mathbb{N} \), the set \( R_s \upharpoonright t \) is finite.
- for all \( t \in \mathbb{N} \), \( R_{s+1} \upharpoonright t \subseteq R_s \upharpoonright t \).
- for all \( t \in \mathbb{N} \), \( W_{s,\text{Id}} \upharpoonright t = (R_s \upharpoonright t)^{F(\bar{x}\upharpoonright t)} \).
$G_\omega$ is the inverse limit of the groups $G_s$ and $G_\omega = F(\bar{x})/W_{\omega, \text{Id}}$
where $W_{\omega, \text{Id}} = \cap W_{s, \text{Id}}$

Need the isomorphism type of $G_\omega$ to be nonrecursive:
Let $A \in \Delta^0_1$ not r.e.
Ensure $n \in A$ iff $\exists l \in \mathbb{N}, g \in G \ g^{f(n,l)} = \text{Id}$ ($f$ computable).

$n \in A$ iff $\lim_{l \to \infty} h(n, l) = 1$ (and for all $n$ this limit exists).

$R_s = \{ x_i^{p_n,l} : \forall j(l \leq j \leq s \ h(n, j) = 1) \}$
s-applying relations

\[ \overline{p} \leftrightarrow \overline{q} \text{ where } (a, b) \in q_i \text{ iff there is a } x_i w' \in W_{s, \text{Id}} \text{ such that } w'(b) = a. \]
Theorem (K)

There exists a noncomputable cofinitary group that has a computable set of generators.