On Hurewicz subsets of $\mathbb{R}^n$

Marion Scheepers

BEST 17
Outline

1. Lusin- and Sierpinski- sets
2. Some dimension-theory concepts
3. Hurewicz sets.
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2. Some dimension-theory concepts
3. Hurewicz sets.
Lusin and Sierpiński sets.

Lusin sets (1913/4)

Definition

$L \subset \mathbb{R}$ is a Lusin set if:

- $|L| = 2^{\aleph_0}$ and
- for each nowhere dense set $N \subset \mathbb{R}$, $|L \cap N| \leq \aleph_0$

Theorem (Lusin-Mahlo)

$ZFC + CH \vdash \text{There is a Lusin set.}$

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Three fundamental results.

Theorem (Rothberger)

\[ \text{ZFC} \vdash \text{CH} \iff \text{there are both a Lusin set and a Sierpiński set}. \]

Theorem (Galvin-Mycielski-Solovay)

If \( L \) is a Lusin set then for each first category set \( M \), \( L+M \neq \mathbb{R} \).

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**Some dimension-theory concepts**

Hurewicz sets.

Two Theorems and Conjecture

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**Notions of countable dimensionality.**

\[ C \subset \mathbb{R}^N \text{ is} \]

**countable dimensional if:**

\( C \) is a union of countably many finite dimensional sets (1928).

**strongly countable dimensional if:**

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Basic Facts.

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\[ \mathbb{R}^N \text{ is not countable dimensional.} \]

**Theorem (Nagami-Smirnov)**

\[ \mathbb{R}^N \text{ is a union of } \aleph_1 \text{ countable dimensional sets.} \]

**Theorem (Folklore)**

There are countable dimensional subsets of \( \mathbb{R}^N \) which are not strongly countable dimensional.
Basic Facts.

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Hurewicz sets (1932)

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A subset $H$ of $\mathbb{R}^N$ is a Hurewicz set if

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If \( H \subset \mathbb{R}^N \) is a Hurewicz set, then \( H + N \neq \mathbb{R}^N \) (indeed, is first category) for each strongly countable dimensional set \( N \subset \mathbb{R}^N \).

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