Reasonable ultrafilters

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Report on works done in cooperation with
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Reasonability

Reasonable ultrafilters on uncountable cardinals were introduced in Shelah [Sh:830] in order to suggest a line of research that would in some sense repeat the beautiful theory created around the notion of $P$–points on $\omega$. The definition of reasonable ultrafilters involves two conditions.

- The first demand, so called the weak reasonability of an ultrafilter, is a way to guarantee that we are not entering the realm of large cardinals: the considered ultrafilter is required to be very non-normal.
- The second part of the definition is a creative re-interpretation of the property that any countable family of members of the ultrafilter has a pseudo-intersection in the ultrafilter.

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1 The combinatorics of reasonable ultrafilters. Fund Math 192 (2006)
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Weak reasonability

Definition: Let $D$ be a uniform ultrafilter on a regular uncountable cardinal $\lambda$. We say that $D$ is weakly reasonable, if for every increasing continuous sequence $\langle \delta_\xi : \xi < \lambda \rangle \subseteq \lambda$ there is a club $C$ of $\lambda$ such that

$$\bigcup \{ [\delta_\xi, \delta_{\xi+1}) : \xi \in C \} \notin D.$$
Ultrafilters from filters on small sets

Let $\mathbb{Q}_\lambda^0$ be the collection of all sequences $r = \langle (\alpha_\xi, d_\xi) : \xi < \lambda \rangle$ such that

- $\langle (\alpha_\xi : \xi < \lambda \rangle$ is an increasing continuous sequence of ordinals below $\lambda$ and
- $d_\xi$ is an ultrafilter on the interval $[\alpha_\xi, \alpha_{\xi+1})$.

For $r \in \mathbb{Q}_\lambda^0$ let $\text{fil}(r)$ be the family of subsets of $\lambda$ which are eventually large in every interval $[\alpha_\xi, \alpha_{\xi+1})$, that is

$$\text{fil}(r) = \{ A \subseteq \lambda : (\exists \zeta < \lambda)(\forall \xi > \zeta)(A \cap [\alpha_\xi, \alpha_{\xi+1}) \in d_\xi) \}.$$  

(The set $\text{fil}(r)$ is a filter on $\lambda$.)

We say that $r \leq^0 s$ if and only if $\text{fil}(r) \subseteq \text{fil}(s)$. ($\leq^0$ is a quasi order on $\mathbb{Q}_\lambda^0$.)


The demand generalizing P-pointness for an ultrafilter $D$ on $\lambda$ is:

\[ (*) \text{ there is a } (\lambda^+)\text{–directed (with respect to } \leq^0) \text{ family } H \]
\[ \text{such that } D = \bigcup \{ \text{fil}(r) : r \in H \}. \]

The family $H$ as above may be called a generating family for the ultrafilter $D$.

Reasonable ultrafilters are ultrafilters which are weakly reasonable and satisfy the condition $(*)$. (So, reasonable ultrafilters are weakly reasonable ultrafilters with $(\lambda^+)\text{–directed generating families.}$)
The two components are connected

**Proposition:** [S.Shelah and AR]
Suppose that $\kappa \leq \lambda$ and $H \subseteq Q_\lambda^0$ is a $(\kappa)$–directed family such that $D := \bigcup \{ \text{fil}(r) : r \in H \}$ is an ultrafilter on $\lambda$. If $D$ is not weakly reasonable, then for some club $C$ of $\lambda$ the quotient ultrafilter $D/C$ is $(\kappa)$–complete and it contains all clubs of $\lambda$. 
There may be reasonable ultrafilters

Theorem: [S. Shelah and AR]

- Assume $\lambda = \lambda^{<\lambda}$ and $\diamondsuit S^\lambda_{\lambda^+}$ holds. There exists a sequence $\langle r_\xi : \xi < \lambda^+ \rangle \subseteq Q^0_\lambda$ such that
  
  (i) $(\forall \xi < \zeta < \lambda^+) (r_\xi \leq^0 r_\zeta)$, and
  (ii) the family $D = \bigcup_{\xi < \lambda^+} \text{fil}(r_\xi)$ is an ultrafilter on $\lambda$ (so it is a reasonable ultrafilter on $\lambda$).

- The forcing notion $Q^0_\lambda = (Q^0_\lambda, \leq^0)$ is $(<\lambda^+)$–complete and $\Vdash_{Q^0_\lambda} \text{“ } G_{Q^0_\lambda} \text{ is a reasonable family generating an ultrafilter } \text{“}$.
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Theorem: [S. Shelah and AR]

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Could there be no reasonable ultrafilters?

**Problem:** Is it consistent that there are no reasonable ultrafilters on \( \lambda \)?

**Theorem:** [S. Shelah and AR]
Assume \( \lambda \) is a strongly inaccessible cardinal. Then there is a forcing notion \( P \) such that

\[ \models_P \text{“} \lambda \text{ is strongly inaccessible and } 2^\lambda = \lambda^{++} \text{ and there is no reasonable ultrafilter on } \lambda \text{ with a generating system of size } < 2^\lambda \text{”} \]
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Small generating systems

**Theorem:** [S.Shelah and AR]
Assume that $\lambda$ is a strongly inaccessible cardinal. Then there is a forcing notion $\mathbb{P}$ such that

$$\mathbb{P} \models \lambda \text{ is strongly inaccessible and } 2^\lambda = \lambda^{++} \text{ and there is a } (\lambda^+)\text{-directed family } H \subseteq Q^0_\lambda \text{ such that } |H| = \lambda^+ \text{ and } \text{fil}(H) \text{ is an ultrafilter on } \lambda, \text{ in particular there is a reasonable ultrafilter on } \lambda \text{ with generating system of size } < 2^\lambda$$
Weak reasonability game

Definition:
Let $D$ be a uniform ultrafilter on $\lambda$. We define a game $\mathcal{D}_D$ between two players, Odd and Even, as follows. A play of $\mathcal{D}_D$ lasts $\lambda$ steps and during a play an increasing continuous sequence $\bar{\alpha} = \langle \alpha_i : i < \lambda \rangle \subseteq \lambda$ is constructed. The terms of $\bar{\alpha}$ are chosen successively by the two players so that Even chooses the $\alpha_i$ for even $i$ (including limit stages $i$ where she has no free choice) and Odd chooses $\alpha_i$ for odd $i$. Even wins the play if and only if $\bigcup \{ [\alpha_{2i+1}, \alpha_{2i+2}) : i < \lambda \} \in D$.

Proposition: [S. Shelah]
Assume $D$ is a uniform ultrafilter on $\lambda$.
- If $D$ is not weakly reasonable, then Odd has a winning strategy in the game $\mathcal{D}_D$.
- If $\lambda$ is strongly inaccessible and Odd has a winning strategy in $\mathcal{D}_D$, then $D$ is not weakly reasonable.
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- If $D$ is not weakly reasonable, then Odd has a winning strategy in the game $\mathcal{D}_D$.
- If $\lambda$ is strongly inaccessible and Odd has a winning strategy in $\mathcal{D}_D$, then $D$ is not weakly reasonable.
Theorem: [S. Shelah and AR]
Assume that there exists a strongly inaccessible cardinal. Then some forcing notion forces that

“there is a $\leq ^*-$increasing sequence $\langle r_\xi : \xi < \omega_2 \rangle \subseteq \mathcal{Q}_\lambda^0$ such that $D := \bigcup (\{ \text{fil}(r_\xi) : \xi < \omega_2 \})$ is a very reasonable ultrafilter on $\omega_1$ but Odd has a winning strategy in the game $\mathcal{D}_D$ ”.