A dichotomy theorem for models of $\text{AD}^+$. 

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Theorem (Caicedo, K., Woodin (2009), Woodin (1995−ε)). Suppose $\text{AD}^+$ together with either $V = L(\mathcal{P}(\mathbb{R}))$ or $V = L(T, \mathbb{R})$ for $T \subseteq \text{OR} \times \mathbb{R}$, then for any set $X$ exactly one of the following hold:

- $|\mathbb{R}| \leq |X|$, i.e., $\exists f : \mathbb{R} \overset{1-1}{\rightarrow} X$.
- $X$ is wellorderable.

Basically this result involves stringing together a few results of H. Woodin. All unattributed theorems and definitions can safely be attributed to Woodin.
For a set $X$, $X^\omega$ is the topological space which is the usual $\omega$-product of $X$ with the discrete topology. So basic open sets are $[s] = \{ f \in X^\omega : s \subseteq f \}$, where $s \in X^{<\omega}$.

In this talk $\mathbb{R}$ will always mean $\omega^\omega$ which is homeomorphic to the irrationals.

A tree $T$ on $X \times Y$ is a subset of $(X \times Y)^{<\omega}$ that is closed under restriction. $[T]$ is the set of infinite branches through $T$. $p[T] = \{ f : \exists g \in Y^\omega (f, g) \in [T] \} = \{ f : T_f \text{ is illfounded} \}$. 

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Basic notions: AD – Suslin sets

\[ A \subseteq X^\omega \] is \( \lambda \)-Suslin iff there is a tree \( T \) on \( X \times \lambda \) such that
\[ A = p[T]. \]

Let \( S_\lambda = \{A \subseteq \mathbb{R} : A \text{ is } \lambda\text{-Suslin}\} \). \( \kappa \) is a Suslin cardinal iff
\[ S_\kappa \setminus \bigcup_{\lambda < \kappa} S_\lambda \neq \emptyset. \]

If \( A \subseteq X^\omega \) is Suslin as given by \( T \subseteq X \times \lambda \), then let \( \phi_i : A \rightarrow \lambda \) be defined by \( \phi_i(f) = b^T_f(i) \) where \( b^T_f \) is the left-most branch of \( T_f \).

The sequence \( \bar{\phi} = \{\phi_i : i \in \omega\} \) is the (semi-)scale associated to \( T \).
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$A \subseteq X^\omega$ is $\lambda$-Suslin iff there is a tree $T$ on $X \times \lambda$ such that $A = p[T]$.

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If \( A \subseteq X^\omega \) is Suslin as given by \( T \subseteq X \times \lambda \), then let \( \phi_i : A \rightarrow \lambda \) be defined by \( \phi_i(f) = b_f^T(i) \) where \( b_f^T \) is the left-most branch of \( T_f \).

The sequence \( \bar{\phi} = \{ \phi_i : i \in \omega \} \) is the (semi-)scale associated to \( T \).
For $A \subseteq X^\omega$, $G_X(A)$ is the infinite perfect information game with two players $I$ and $II$, where the players take turns playing elements of $X$ in $\omega$-many moves and cooperate to build $f \in X^\omega$. Player $I$ wins if $f \in A$, otherwise player $II$ wins.

A winning strategy for $I$ in $G_X(A)$ is a function $\sigma : X^{<\omega} \rightarrow X$ so that $\sigma \ast f = \langle \sigma(\emptyset), f(0), \sigma(\langle f(0) \rangle), f(1), \sigma(\langle f(0), f(1) \rangle), \ldots \rangle$ is always a win for $I$. Similarly for $II$. $A \subset X^\omega$ is **determined** if one of the players has a winning strategy.

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AD implies that all sets have the perfect set property, are measurable, have the property of Baire, etc. In particular choice fails under AD, so AD is only considered in certain inner models, like $L(\mathbb{R})$.

Assuming certain large cardinals exist, AD holds in $L(\mathbb{R})$ (in all generic extensions of $V$). Moreover, $AD^{L(\mathbb{R})}$ in all generic extensions of $V$ yields inner models for large cardinals.
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AD$^+$ is a “strengthening” of AD. The name and most of the structure theory for models of AD$^+$ are due to H. Woodin. AD$^+$ is intended to capture the theory of a model $M$ of AD all of whose sets of reals are Suslin (have scales) in some larger model $N$ of AD, hence the notion was originally termed *within scales*.

**Definition.** AD$^+$ is AD + DC$_R$ together with

- All sets of reals are $\infty$-Borel.
- $<$\(\Theta\)-ordinal determinacy.
Basic Notions: $\text{AD}^+$

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**Definition.** $\text{AD}^+$ is $\text{AD} + \text{DC}_R$ together with

- All sets of reals are $\infty$-Borel.
- $\langle \Theta \rangle$-ordinal determinacy.
Essentially the $\infty$-Borel sets is what you get by extending the usual Borel hierarchy by allowing arbitrary wellordered unions.

Without choice you work with codes for sets rather than the sets themselves, hence an $\infty$-Borel set is any set with an $\infty$-Borel code.

There are many, essentially equivalent, definitions of $\infty$-Borel codes; one of them is as follows:

- An $\infty$-Borel code is a pair $(\phi, S)$ where,
  - $\phi$ is a $\Sigma_1$-formula (of the language of set theory) with two free variables and
  - $S$ is a set of ordinals.
- $A(\phi, S) = \{ x : L[S, x] \models \phi(S, x) \}$ is the set coded.

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For example, if $A$ is Suslin as witnessed by the tree $T$ on $\omega \times \kappa$, then

$$x \in A \text{ iff } L[T, x] \models T_x \text{ is wellfounded}$$

So Suslin sets are $\infty$-Borel.

Suslin subsets of $\mathbb{R} \times \mathbb{R}$ can be uniformized, a fragment of choice, and in general there can be non-uniformizable sets in a model of $\mathsf{AD}$, so it is not true that all $\infty$-Borel sets are Suslin.
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Under AD, there are undetermined ordinal games, namely, player I plays \( \alpha < \omega_1 \) and player II plays bit by bit a real coding \( \alpha \). Clearly, any play by I can be defeated by II, but II cannot have a winning strategy since this would yield an uncountable sequence of reals.

However, also under AD, any Suslin/co-Suslin subset of \( \lambda^\omega \) for any \( \lambda < \Theta \) is determined. So if \( A \subseteq \mathbb{R} \) is Suslin/co-Suslin and \( f : \lambda^\omega \to \mathbb{R} \) is continuous, then the induced game \( G\lambda(f^{-1}[A]) \) is determined.

**Definition.** **Ordinal determinacy** is the assertion that for all \( \lambda < \Theta \), for all \( f : \lambda^\omega \to \mathbb{R} \) continuous, and for all \( A \subseteq \mathbb{R} \), the induced game on \( \lambda \) is determined.
Basic Notions: $\text{AD}^+ –$ Ordinal determinacy

Under $\text{AD}$, there are undetermined ordinal games, namely, player $I$ plays $\alpha < \omega_1$ and player $II$ plays bit by bit a real coding $\alpha$. Clearly, any play by $I$ can be defeated by $II$, but $II$ cannot have a winning strategy since this would yield an uncountable sequence of reals.

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Basic Notions: AD$^+$– Ordinal determinacy

One consequence of ordinal determinacy that we will use is the following:

**Theorem.** For every Suslin cardinal $\kappa$, there is a unique normal fine measure $\mu_\kappa$ on $\mathcal{P}_{\omega_1}(\kappa)$. In particular $\mu_\kappa \in \text{OD}$.

For any $\gamma < \kappa$ where $\kappa$ is Suslin define $\mu_\gamma = \pi_{\kappa,\gamma}(\mu_\kappa)$ where $\pi_{\kappa,\gamma} : \mathcal{P}_{\omega_1}(\kappa) \to \mathcal{P}_{\omega_1}(\gamma)$ is defined by $\sigma \mapsto \sigma \cap \gamma$.

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If $N \subseteq M$ are models of AD with the same reals and every set of reals in $N$ is Suslin in $M$, then $N \models AD^+$. So the original notion of within scales implies AD$^+$.

In the other direction, if $N \supseteq OR$ is a model of AD$^+$, then there is a generic extension of an inner model of $N$ such that every $A \in \mathcal{P}(R) \cap N$ is Suslin in some $\omega$-model $M$ which also satisfies AD and which has the same reals as $N$.

So the axiomatization captures what it was supposed to.
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So the axiomatization captures what it was supposed to.
Let $\kappa_\infty = \sup \{ \kappa : \kappa \text{ is a Suslin cardinal} \}$. AD implies that the Suslin cardinals are closed below $\kappa_\infty$.

$AD_\mathbb{R}$ is equivalent to $\kappa_\infty = \Theta$.

$AD^+$ is equivalent to $AD + DC_\mathbb{R}$ together with “the Suslin cardinals are closed below $\Theta$”.

Thus if there is a model of $AD + \neg AD^+$, then in this model $\kappa_\infty < \Theta$ and $\kappa_\infty$ is not a Suslin cardinal.
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Assuming $\mathbf{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$, if $\kappa_\infty < \Theta$ and $T \subseteq \omega \times \kappa_\infty$ is a tree witnessing $\kappa_\infty$ is Suslin, then $V = L(T^*, \mathbb{R})$ where $T^* = \prod_x T/\mu_T$, and where $\mu_T$ is the $T$-degree Martin measure.

Thus assuming $\mathbf{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$, either $V$ is a model of $\mathbf{AD}_\mathbb{R}$ or $V = L(T, \mathbb{R})$ for some set of ordinals $T$.

Under $\mathbf{AD}_\mathbb{R} + V = L(\mathcal{P}(\mathbb{R}))$ we have $V = \text{OD}((<\Theta)^\omega)$, where $(<\Theta)^\omega = \bigcup_{\gamma < \Theta} \gamma^\omega$. 

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There are a couple of ways that $\infty$-Borel codes are “close” to the set coded. One way is expressed by the following fact.

**Theorem.** Assume $\text{AD}$, then given an $\infty$-Borel set $A$, there is an $\infty$-Borel code for $A$ which is $\Sigma^1_3(A)$.

More relevant to us is the following:

**Theorem.** Assume $\text{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$. Let $T \subseteq \text{OR}$ and let $A \subseteq \mathbb{R}$ be $\text{OD}_T$, then $A$ has an $\text{OD}_T$ $\infty$-Borel code.
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Just as an example of how determinacy can be separated from its structural consequences, the preceding theorem essentially is proved by showing:

**Theorem.** Suppose $V = L(\mathcal{P}(\gamma)) \models ZF + DC$ and $\mu$ is a fine measure on $\mathcal{P}_{\omega_1}(\mathcal{P}(\gamma))$ in $V$, then for all $T \subseteq OR$ and $A \subseteq \mathbb{R}$, if $A \in OD_{T,\mu}$, then $A$ is $\infty$-Borel with $\infty$-Borel code in $HOD_{T,\mu}$.

**Fact.** Under $AD$ there is an $OD$ measure on $\mathcal{P}_{\omega_1}(\mathcal{P}(\gamma))$ for all $\gamma < \Theta$.

So as a corollary, if $AD^+$ holds and $V = L(\mathcal{P}(\gamma))$ for $\gamma < \Theta$ and $A \in OD_T \cap \mathcal{P}(\mathbb{R})$, then $A$ has $OD_T \infty$-Borel code.

This almost gives the theorem from the previous slide since assuming $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$ we have $V = L(\bigcup_{\gamma < \Theta} \mathcal{P}(\gamma))$. 
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The dichotomy: Closeness of code to set

Just as an example of how determinacy can be separated from its structural consequences, the preceding theorem essentially is proved by showing:

**Theorem.** Suppose $V = L(\mathcal{P}(\gamma)) \models \text{ZF + DC}$ and $\mu$ is a fine measure on $\mathcal{P}_{\omega_1}(\mathcal{P}(\gamma))$ in $V$, then for all $T \subseteq \text{OR}$ and $A \subseteq \mathbb{R}$, if $A \in \text{OD}_{T,\mu}$, then $A$ is $\infty$-Borel with $\infty$-Borel code in $\text{HOD}_{T,\mu}$.

**Fact.** Under $\text{AD}$ there is an OD measure on $\mathcal{P}_{\omega_1}(\mathcal{P}(\gamma))$ for all $\gamma < \Theta$.

So as a corollary, if $\text{AD}^+$ holds and $V = L(\mathcal{P}(\gamma))$ for $\gamma < \Theta$ and $A \in \text{OD}_T \cap \mathcal{P}(\mathbb{R})$, then $A$ has $\text{OD}_T$ $\infty$-Borel code.

This almost gives the theorem from the previous slide since assuming $\text{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$ we have $V = L(\bigcup_{\gamma < \Theta} \mathcal{P}(\gamma))$. 
The dichotomy: The $L(T, \mathbb{R})$ case.

Theorem. Suppose $V = L(T, \mathbb{R}) \models ZF + DC_\mathbb{R}$ and $\mu$ is a fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$, then every set is $\infty$-Borel.

Main idea... All you need to do is come up with a way of getting an $\infty$-Borel code for $\exists^\mathbb{R} A_S$ from $S$. The key fact is:

$$\exists y A_S(x, y) \iff \forall^* \sigma \text{ HOD}^{L[T,S,\sigma]}_T[x] \text{Col}(\omega, \Theta^{L[T,S,\sigma]}_L) \models \exists y A_S(x, y)$$

We can then form an ultrapower to get

$$\langle H^\infty, S^\infty, \theta^\infty \rangle = \prod_\sigma \langle \text{HOD}^{L[T,S,\sigma]}_T, S, \Theta^{L[T,S,\sigma]}_L \rangle$$

and

$$\exists y A_S(x, y) \iff H^\infty[x]^{\text{Col}(\omega, \theta^\infty)} \models \exists y A_{S^\infty}(x, y)$$

The RHS is an $\infty$-Borel code for $\exists^\mathbb{R} A_S$. 

Ketchersid A dichotomy theorem for models of $\text{AD}^+$. 
The dichotomy: The $L(T, \mathbb{R})$ case.

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$$\exists y A_S(x, y) \text{ iff } \forall^*_\mu \sigma \ HOD_{L[T,S,\sigma]}^L[x] \text{Col}(\omega, \Theta_{L[T,S,\sigma]}) \models \exists y A_S(x, y)$$

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Ketchersid A dichotomy theorem for models of AD$^+$. 

The dichotomy: The \( L(T, \mathbb{R}) \) case.

An argument similar to the preceding one shows that if \( E \) is thin, i.e., \( |\mathbb{R}| \not\subseteq \mathbb{R}/E \), then \( \mathbb{R}/E \) is wellorderable.

The idea here is also fairly simple. Let \( S \) be an \( \infty \)-Borel code for \( E \). Let \( Q^\sigma = B_{\infty}^{\text{HOD}}_{T,S} / \sim^\sigma \) where

\[
S_0 \sim^\sigma S_1 \text{ iff } (A_{S_0} = A_{S_1})^{L[T,S,\sigma]}
\]

So \( Q^\sigma \) is the \( \infty \)-Borel version of the Vopenka algebra in \( L[T,S,\sigma] \).

Let \( Q^\infty = \prod_\sigma Q^\sigma \), this the \( \infty \)-Borel algebra of \( H^\infty \) modulo \( \sim^\infty \). It is not difficult to check that \( S_0 \sim^\infty S_1 \text{ iff } (A_{S_0} = A_{S_1})^{L(T,\mathbb{R})} \).
The dichotomy: The $L(T, \mathbb{R})$ case.

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$$S_0 \sim^\sigma S_1 \iff (A_{S_0} = A_{S_1})^{L[T,S,\sigma]}$$

So $Q^\sigma$ is the $\infty$-Borel version of the Vopenka algebra in $L[T,S,\sigma]$.

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Ketchersid  A dichotomy theorem for models of $\text{AD}^+$. 
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Ketchersid A dichotomy theorem for models of $\text{AD}^+$. 
There are two cases.

Either $D$ is dense where

$$D = \{ c \in \mathbb{Q}^\infty : (c, c) \Vdash_{\mathbb{Q}^\infty \times \mathbb{Q}^\infty} \dot{r}_0 \ E_{\mathbb{S}^\infty} \ \dot{r}_1 \}$$

In this case each $E$-class is “captured” by an $\infty$-Borel set with code in $\mathbb{Q}^\infty$ and we get a canonical wellordering of $\mathbb{R}/E$ of ordertype $<(\theta^\infty)^+$.
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Ketchersid A dichotomy theorem for models of $\AD^+$. 
An initial dichotomy

Otherwise the set $D$ is not dense. In this case get $c \neq 0_{\mathbb{Q}}$ such that for every $c' \leq_{\mathbb{Q}} c$, there are $c_0, c_1 \leq_{\mathbb{Q}} c'$ such that

$$(c_0, c_1) \models_{\mathbb{Q}_\infty \times \mathbb{Q}_\infty} \dot{r}_0 \mathcal{E}_{\mathbb{Q}} \dot{r}_1$$

We can then find a measure one set of $\sigma$ and $c^\sigma \in \mathbb{Q}^\sigma$ so that $c^\sigma \neq 0_{\mathbb{Q}^\sigma}$ and

$$\forall c' \leq_{\mathbb{Q}^\sigma} \exists c_0, c_1 \leq_{\mathbb{Q}^\sigma} (c_0, c_1) \models_{\mathbb{Q}^\sigma \times \mathbb{Q}^\sigma} \dot{r}_0 \mathcal{E}_{\mathbb{Q}} \dot{r}_1$$

Form this we can, in $V$, build a perfect set of $E$-inequivalent reals.
An initial dichotomy

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Form this we can, in $V$, build a perfect set of $E$-inequivalent reals.
The dichotomy: The $L(T, \mathbb{R})$ case.

The following also holds:

**Theorem.** Assume $V = L(T, \mathbb{R}) \models \text{ZF} + \text{DC}_\mathbb{R}$ and $\mu$ is a fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$. If $A \subset \mathbb{R}$ is $\text{OD}_{T,r}$, then $A$ has an $\text{OD}_{T,r,\mu}$ $\infty$-Borel code.

Thus we have:

**Theorem.** Assume $V = L(T, \mathbb{R}) \models \text{ZF} + \text{DC}_\mathbb{R}$ and there is a fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$, then if $E \in \text{OD}_{T,r}$ is a thin equivalence relation, there is $S \in \text{OD}_{T,r,\mu}$, an $\infty$-Borel code for $E$, and $\phi : \mathbb{R}/E \to \gamma$ in $\text{OD}_{T,r,\mu}$, computed from $S$, such that $x E y$ iff $\phi(x) = \phi(y)$. In particular $\mathbb{R}/E \subset \text{OD}_{T,r,\mu}$.
The dichotomy: The $L(T, \mathbb{R})$ case.

Assume $V = L(T, \mathbb{R}) \models ZF + DC_{\mathbb{R}}$ and there is a fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$, let $X$ be any set, then $X$ is $OD_{T,r}$ for some real $r$. For each $\alpha$ set

$$X_\alpha = \{ a \in X : \exists t \in \mathbb{R} \ [a \text{ is definable using } r, t, \mu, \alpha] \}$$

Let $E_\alpha$ be the corresponding equivalence relation on $\mathbb{R}$ and let $S_\alpha$ be an $OD_{T,r,\mu}$ $\infty$-Borel code for $E_\alpha$. If each $E_\alpha$ is thin, then we get, uniformly in $\alpha$, $\phi_\alpha : \mathbb{R}/E_\alpha \to \gamma_\alpha$ inducing $E_\alpha$.

One nice way of putting this is:

**Theorem.** If $X \in OD_{T,r}$ is thin, then $X \subset OD_{T,r,\mu}$. Hence $X$ is wellorderable.
Assume $V = L(T, \mathbb{R}) \models ZF + DC_{\mathbb{R}}$ and there is a fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$, let $X$ be any set, then $X$ is $OD_{T,r}$ for some real $r$. For each $\alpha$ set

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One nice way of putting this is:

Theorem. If $X \in OD_{T,r}$ is thin, then $X \subset OD_{T,r,\mu}$. Hence $X$ is wellorderable.
The dichotomy: The $\text{AD}_\mathbb{R}$ case

Assume $\text{AD}^+$ and $V \neq L(T, \mathbb{R})$, then $V = L(\mathcal{P}(\mathbb{R}))$ and hence $V = \text{OD}((<\Theta)\omega)$.

Let $X \in \text{OD}_{s_0}$ for some $s_0 \in \gamma\omega$ for some $\gamma < \Theta$.

For $\sigma \in [<\Theta]^{\omega}$ define

$$X_{\sigma,\alpha} = \{a \in X : \exists t \in \mathbb{R} \ [a \text{ is definable from } \sigma, s_0, \alpha, t] \}$$

Notice that if $\sigma \subset \tau$ and $a \in \text{OD}_{\sigma,s_0,t}$ for some $t$, then there is $t' \in \mathbb{R}$ so that $a \in \text{OD}_{\tau,s_0,t'}$.

The reason for relativizing to $\sigma$ will become apparent soon.

Let $E_{\sigma,\alpha}$ be the corresponding equivalence relation on $\mathbb{R}$. If any $E_{\sigma,\alpha}$ is thick, then we are done.
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Assume $\text{AD}^+$ and $V \neq L(T, \mathbb{R})$, then $V = L(\mathcal{P}(\mathbb{R}))$ and hence $V = \text{OD}((<\Theta)^\omega)$.

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Assume $AD^+$ and $V \neq L(T, \mathbb{R})$, then $V = L(\mathcal{P}(\mathbb{R}))$ and hence $V = OD((<\Theta)^\omega)$.

Let $X \in OD_{s_0}$ for some $s_0 \in \gamma^\omega$ for some $\gamma < \Theta$.

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Notice that if $\sigma \subset \tau$ and $a \in \text{OD}_{\sigma, s_0, t}$ for some $t$, then there is $t' \in \mathbb{R}$ so that $a \in \text{OD}_{\tau, s_0, t'}$.

The reason for relativizing to $\sigma$ will become apparent soon.

Let $E_{\sigma,\alpha}$ be the corresponding equivalence relation on $\mathbb{R}$. If any $E_{\sigma,\alpha}$ is thick, then we are done.
Otherwise, uniformly in $\alpha$, there is an $\text{OD}_{\sigma,s_0}$ $\infty$-Borel code $S_{\sigma,\alpha}$ for $E_{\sigma,\alpha}$ and corresponding $\phi_{\sigma,\alpha} : \mathbb{R}/E_{\sigma,\alpha} \to \gamma_\alpha$ inducing $E_{\sigma,\alpha}$.

In particular $X_{\sigma,\alpha} \subset \text{OD}_{\sigma,s_0}$ and thus $X_\sigma \subset \text{OD}_{\sigma,s_0}$, where $X_\sigma = \bigcup_\alpha X_{\sigma,\alpha}$.

Let $<_\sigma$ be the $\text{OD}_{\sigma,s_0}$ wellorder of $X_\sigma$. 

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Ketchersid A dichotomy theorem for models of $\text{AD}^+$. 
The dichotomy: The \( \text{AD}_R \) case

For each \( \xi < \Theta \) let \( X_\xi = \bigcup_{\sigma \in \mathcal{P}_{\omega_1}(\xi)} X_\sigma \).

Notice for \( \xi < \xi' \), \( X_\xi \subseteq X_{\xi'} \).

Woodin’s main observation here was that the supercompactness measures can be used to uniformly wellorder the \( X_\xi \) and hence we get a wellorder of \( X \). Namely,

\[
a <_\xi a' \iff \forall^*_{\mu_\xi} \sigma [a <_{\sigma} a']
\]

This gives \( X_\xi \subset \text{OD}_{s_0} \) and hence \( X \subset \text{OD}_{s_0} \). So \( X \) is wellorderable.
The dichotomy: The $\text{AD}_\mathbb{R}$ case

For each $\xi < \Theta$ let $X_\xi = \bigcup_{\sigma \in \mathcal{P}_{\omega_1}(\xi)} X_\sigma$.

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This gives $X_\xi \subset \text{OD}_{s_0}$ and hence $X \subset \text{OD}_{s_0}$. So $X$ is wellorderable.
The dichotomy: The $\mathbf{AD}_R$ case

For each $\xi < \Theta$ let $X_\xi = \bigcup_{\sigma \in \mathcal{P}_{\omega_1}(\xi)} X_\sigma$.

Notice for $\xi < \xi'$, $X_\xi \subseteq X_{\xi'}$.

Woodin’s main observation here was that the supercompactness measures can be used to uniformly wellorder the $X_\xi$ and hence we get a wellorder of $X$. Namely,

$$a <_{\xi} a' \text{ iff } \forall^* \mu \in (\xi) \ [a <_{\sigma} a']$$

This gives $X_\xi \subset \mathbf{OD}_{s_0}$ and hence $X \subset \mathbf{OD}_{s_0}$. So $X$ is wellorderable.
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For each $\xi < \Theta$ let $X_\xi = \bigcup_{\sigma \in \mathcal{P}_{\omega_1}(\xi)} X_\sigma$.

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