MATH 497 / 597 – Homework 5

Your work on this assignment must be handed in by Tuesday, 29 October 2002 at 3:15 p.m. GOOD LUCK!

1) Write code to solve the heat equation (8.2.1) with initial condition (8.2.2) and boundary conditions (8.2.3) via the explicit method (8.2.4) and the implicit method (8.3.2). When you are done, send your codes to me via email attachments.

2) In class we derived the stability condition for the explicit method (8.2.4) for the heat equation (8.2.1). Use the code you wrote above to implement the explicit method for heat equation using the following values:

- $c = 0.5$
- $n = 9$
- $\Delta t = 0.02$
- $\alpha = \beta = 0$
- $g(x) = x(1 - x)$

a) Show that the stability condition is violated if we choose the values given immediately above.

b) Run your code until you see instability. You should see it pretty clearly after only 15 time steps. Hand in a hard copy of your run.

c) While keeping all other values the same, change Δt so that the stability condition is satisfied. Run the code again to see that the unstable behavior seen in part b) has disappeared. Hand in a hard copy of this run.

d) Write a sentence or two which completely yet succinctly describes the phenomenon of instability.

3) Describe two different techniques for modifying the implementation of the implicit method (8.3.2) for solving the heat equation if we change the right hand boundary condition from $u(t, 1) = \beta$ to $\dfrac{\partial u}{\partial x}(t, 1) = \gamma$. Your description should include writing out the appropriate matrix equations and discussing the advantages and disadvantages of each of your two techniques.

4) Do Exercise 8.3.5 on page 266.

5) **Required for graduate students only.** As noted in the text, the Crank-Nicolson method, when applied to the heat equation, may be written as (8.3.13). If the matrix

$$I + \frac{\mu}{2} A \qquad (1)$$

is not singular, then we may write (8.3.13) as

$$\mathbf{u}^{m+1} = \left(I + \frac{\mu}{2}A\right)^{-1} \left(I - \frac{\mu}{2}A\right) \mathbf{u}^m + \left(I + \frac{\mu}{2}A\right)^{-1} \mathbf{b}. \tag{2}$$

It is not difficult to see (although we will not dwell on the details here) that the iteration (2) is stable if and only if the magnitude of all eigenvalues of

$$\left(I + \frac{\mu}{2}A\right)^{-1} \left(I - \frac{\mu}{2}A\right) \tag{3}$$

is less than unity.

a) Let ρ be an eigenvalue of *any* $n \times n$ matrix A. Let $\mathbf{x}$ be an eigenvector of A associated with the eigenvalue ρ. Prove that

$$\frac{1 - \frac{\mu}{2}\rho}{1 + \frac{\mu}{2}\rho}$$

is an eigenvalue of (3) with associated eigenvector $\mathbf{x}$. (You may assume that (1) is nonsingular.)

b) Now, for the case where A = tridiag(-1, 2, -1), find all eigenvalues and eigenvectors of (3).

c) How do your results in parts a) and b) relate to your work above on Exercise 8.3.5?