Math 187 Final Exam

Dr. Holmes

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The final exam begins at 10:30 am and officially ends at 12:30 pm. The exam will continue until 12:45 pm unless someone objects (all must agree). You may use any calculator and you may use your textbook, any class handouts (including old homework and tests) and any notes in your own handwriting. The use of other resources is forbidden. Cell phones and PDA’s may not be used as calculators; cell phones must be turned off and out of sight.
1. Demonstrate that 

\[ P \rightarrow (Q \rightarrow R) \]

and

\[ (P \land Q) \rightarrow R \]

are logically equivalent by constructing a truth table. Be sure to explain why the truth table verifies the logical equivalence.
2. Determine whether the equation

\[ A \cup (B - C) = (A \cup B) \cap (A - C) \]

is an identity or not by drawing Venn diagrams. Be sure to provide a detailed key to your diagrams and highlight the sets in both diagrams which are being shown to be equal (or not equal, as the case may be).
3. (a) Draw a diagram which shows why the circle $x^2 + y^2 = 1$ (center at the origin, radius 1) and the circle $x^2 + y^2 = 4$ (center at the origin, radius 2) have the same infinite cardinality (the two sets of points are the same size). The bijection is very easy to picture geometrically!

(b) Write down the three statements which must be proved to show that the relation “$A$ is the same size as $B$” between sets $A$ and $B$ is an equivalence relation, without using the words “reflexive”, “symmetric”, or “transitive” (your answer should make it clear that you know what these words mean). You do not need to prove the statements!
4. (a) Use the Euclidean algorithm to find the gcd of 65 and 144, showing all calculations on your page. Show how to use the gcd calculation to express \( \gcd(65, 144) \) in the form \( 65m + 144n \).

(b) Give the solution of the equation

\[
65x \equiv 1 \mod 144
\]

(c) (Chinese remainder theorem) Give the smallest positive solution to the system of equations

\[
x \equiv 12 \mod 65
\]

\[
x \equiv 22 \mod 144
\]
5. Prove by mathematical induction that the sum

\[ \sum_{i=0}^{n} 2^i = 1 + 2 + \ldots + 2^{n-1} + 2^n \]

is equal to \(2^{n+1} - 1\) for any positive integer \(n\).
6. Counting problems:

The license plates for a small state consist of three letters followed by three digits (0-9). State how many possible plates there are under each of the following conditions. Show calculations that make it clear how you obtain your answer.

(a) Three letters followed by three digits; no other restrictions.

(b) Three letters and three digits, intermingled in any order.

(c) Three letters followed by three digits, no letter or digit appearing more than once.

(d) Three letters followed by three digits, no letter or digit immediately followed by the same letter or digit.

(e) Three letters followed by three digits, containing exactly one A and exactly one 8.
7. In each part, draw graphs with required degree sequence or explain why such a graph is impossible.

(a) 5,4,3,2,1,1

(b) 5,3,3,2,2,2

(c) 5,4,3,3,2,2,2
8. Planar graphs.

(a) Redraw the indicated graph so as to show that it is a planar graph.

(b) A connected planar graph with six vertices and eight edges divides the plane into how many regions? Show a calculation which verifies this (using the appropriate formula), and draw a picture of such a graph.

(c) Explain why a graph with no more than five vertices, each vertex having degree no greater than three, must be a planar graph. Do not leave details to my imagination in your explanation.
9. Show the steps of sorting the list 3,2,1,5,4,3, first using bubble sort, then using merge sort.
10. (Extra Credit) Since I threatened you with this, you may earn extra credit by giving a full explanation why the longest path in a connected graph must be a Hamiltonian cycle (when the first vertex is copied at the end) if the first and last vertex in the path are linked by an edge. Don’t attempt this until finished with the rest.