Math 502 Homework 4

Randall Holmes

February 27, 2008

This will be due on Tuesday the 11th of March. Material on this homework is fair game for the exam on Thursday the sixth of March!

1. This question relies on ordinary knowledge about the reals and the rationals, and also knowledge of Lebesgue measure if you have studied this (if you haven't, don't worry about that part of the question).

   Verify that the relation on real numbers defined by "$x\,R\,y$ iff $x - y$ is rational" is an equivalence relation.

   Describe the equivalence classes under this relation in general. Describe two or three specific ones. Note that each of the equivalence classes is countably infinite (why?), distinct equivalence classes are disjoint from each other, and so we "ought" to be able to choose a single element from each class.

   Can you think of a way to do this (you will not be able to find one, but thinking about why it is difficult is good for you)?

   Suppose we had a set X containing exactly one element from each equivalence class under R. For each rational number q, let X_q be the set $\{r + q \mid r \in X\}$. Note that X_q is just a translation of X.

   Prove that $\{X_q \mid q \in \mathbb{Q}\}$ is a partition of $\mathbb{R}$. (This will include a proof that the union of the X_q's is the entire real line).

   If you know anything about Lebesgue measure, you might be able to prove at this point that X is not Lebesgue measurable (if you can, do so). It is useful to note that the collection of X_q's is countable.

2. Suppose that P is a partition.

   Prove that the relation $\sim_P$ defined by
   $x \sim_P y$ iff $(\exists A \in P. x \in A \wedge y \in A)$
   is an equivalence relation. What is the field of this equivalence relation?

   Describe its equivalence classes.

   This is an exercise in carefully writing everything down, so show all details of definitions and proof strategy, as far as you can.

3. Prove that $|\mathbb{N}| + 1 = |\mathbb{N}| + |\mathbb{N}| = |\mathbb{N}| \cdot |\mathbb{N}| = |\mathbb{N}|$.

   Describe bijections by arithmetic formulas where you can; in any case clearly describe how to construct them (these are all familiar results, or should be, and all of the bijections can in fact be described algebraically: the formula for triangular numbers can be handy for this). I'm looking for bijections with domain $\mathbb{N}$ and range some more complicated set in every case.

4. Verify the distributive law of multiplication over addition in cardinal arithmetic,
   $$|A| \cdot (|B| + |C|) = |A| \cdot |B| + |A| \cdot |C|,$$
   by writing out explicit sets with the two cardinalities (fun with cartesian products and labelled disjoint unions!) and explicitly describing the bijection sending one set to the other. You do not need to prove that it is a bijection: just describe the sets and the bijection between them precisely.