This is the second Math 502 homework assignment.

Now due Thursday Feb 14: please come and see me at least once with every homework assignment. I will want to see drafts of your work to make sure it is heading in the right direction (and to detect errors in my presentation). Hints for all of these are available on request.

1. Write a definition of the natural number 2 in the form \( \{ x \mid \phi[x] \} \) where \( \phi \) is a formula containing only variables, logical symbols, equality and membership. Hint: the formula \( \phi[x] \) needs to express the idea that \( x \) has exactly two elements in completely logical terms. I will give a definition of 1 with these properties in class.

2. The usual definition of the ordered pair (which we will discuss shortly but which we will not use until we get to section 4) is the following (due to Kuratowski):

\[
\langle x, y \rangle = \text{def} \{ \{ x \}, \{ x, y \} \}.
\]

We will not use this as our definition of ordered pair because it has the inconvenient feature that the pair is two types higher than its projections. What we can do (as an exercise in thinking about sets) is prove the following basic Theorem about this pair definition:

\[
\langle x, y \rangle = \langle z, w \rangle \rightarrow x = z \land y = w
\]

This is your exercise. There are various ways to approach it: one often finds it necessary to reason by cases. if you have seen a proof of this, don’t go look it up: write your own.

3. Prove that the set \( \mathbb{N}^{k+3} \) (the set of natural numbers in type \( k+3 \)) is inductive. You don’t need to specify types on every variable (or constant) every time it occurs, but you might want to state the type of each object mentioned in the proof the first time it appears.

This proof is among other things an exercise in the careful reading of definitions.

4. Prove the following statement using the Peano axioms in the form stated in the current section: \( (\forall n \in \mathbb{N}. n = 0 \lor (\exists m. m + 1 = n)) \). You will need to use induction (use the proofs in Peano arithmetic I
did in class as a model – but this theorem is extremely easy to prove once it is set up).

Why is the object $m$ unique in case it exists? (This is a throwaway corollary of the main theorem: it does not require an additional induction argument).