Math 333 Worksheet (with solutions)

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For each problem, write your final answer as a piecewise defined function without using Heaviside functions.

This is now complete. It is quite likely that errors have crept in in the course of the rather complicated typesetting process, but the general approach should be clear.

1.

$$y'' + 3y' + 2y = f(t), y(0) = 1, y'(0) = 0$$

where $f(t) = t$ for $t < 2$

and $f(t) = 2$ for $t \geq 2$

Solution:

The piecewise function expressed in terms of Heaviside functions:

$$f(t) = t + H(t-2)(2-t)$$

The equation to be solved for the Laplace transform (what you get when you take the Laplace transform of the original equation):

$$s^2L\{y\} - s + 3sL\{y\} - 3 + 2L\{y\} = \frac{1}{s^2} - \frac{e^{-2s}}{s^2}$$

The expression for the Laplace transform of y:

$$\frac{s^3 + 3s^2 + 1 - e^{-2s}}{s^2(s+1)(s+2)}$$

The partial fraction expansion:

$$\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s+1}+\frac{D}{s+2}=\frac{s^3+3s^2+1}{s^2(s+1)(s+2)}$$

$$A(s)(s+1)(s+2)+B(s+1)(s+2)+Cs^2(s+2)+Ds^2(s+1)=s^3+3s^2+1$$

plug in first $s=0, s=-1, s=-2$ to get B, C, D, then any other value (I used $s=1$) to solve for A using the values of B, C, D already computed.

solution $A=-\frac{3}{4}, B=\frac{1}{2}, C=3, D=-\frac{5}{4}$

for the part without Heaviside functions

and

$$A(s)(s+1)(s+2)+B(s+1)(s+2)+Cs^2(s+2)+Ds^2(s+1)=-1$$

solution

$$A=\frac{3}{4}, B=-\frac{1}{2}, C=-1, D=\frac{1}{4}$$

for the part with Heaviside functions.

The inverse Laplace transform of $\frac{-\frac{3}{4}}{s}+\frac{\frac{1}{2}}{s^2}+\frac{3}{s+1}+\frac{-\frac{5}{4}}{s+2}$ is $-\frac{3}{4}+\frac{1}{2}t+3e^{-t}-\frac{5}{4}e^{-2t}$.

The inverse Laplace transform of $e^{-2s}(\frac{\frac{3}{4}}{s}+\frac{-\frac{1}{2}}{s^2}+\frac{-1}{s+1}+\frac{\frac{1}{4}}{s+2})$ is $H(t-2)(\frac{3}{4}-\frac{1}{2}(t-2)-e^{t-2}+\frac{1}{4}e^{-2(t-2)})$.

When all terms are collected, we find that for $t<2$, $y=-\frac{3}{4}+\frac{1}{2}t+3e^{-t}-\frac{5}{4}e^{-2t}$, and for $t>2$, $y=1+(3-e^2)e^{-t}+(\frac{e^4-5}{4})e^{-2t}$.

2.

$$y''+4y=g(t), y(0)=0, y'(0)=-1$$

where $g(t)=\sin(t)$ for $t<2\pi$

and $g(t)=0$ for $t\geq 2$

The solution I gave in class was mostly correct, but I think there was a slight error at the end.

the function $g(t) = \sin(t) - H(t-2\pi)\sin(t)$

$$L\{H(t-2\pi)\sin(t)\} = e^{-2\pi s}L\{\sin(t+2\pi)\} = e^{-2\pi s}L\{\sin(t)\} = \frac{e^{-2\pi s}}{s^2+1}$$

The equation to solve for $L\{y\}$ is

$$s^2L\{y\} + 1 + 4L\{y\} = \frac{1}{s^2+1} - \frac{e^{-2\pi s}}{s^2+1}$$

and when you solve it algebraically you should get

$$L\{y\} = \frac{-s^2 - e^{-2\pi s}}{(s^2+1)(s^2+4)}$$

The partial fraction decomposition should look like this

$$\frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

and the equations to be solved look like this

$$(As+B)(s^2+4) + (Cs+D)(s^2+1) = -s^2$$

for the part without Heaviside functions

and

$$(As+B)(s^2+4) + (Cs+D)(s^2+1) = -1$$

for the part with the Heaviside function.

I show a different way to solve these. We can use $s = i$ and $s = 2i$ to solve these, since these are zeroes of s^2+1 and s^2+4. I'll show how to solve the first one this way, and just give the solution for the second.

$$s = i:$$

$$(Ai + B)(3) + (Ci + D)(0) = 1$$

so $3A = 0$ (there is no imaginary part) and $3B = 1$, so $B = \frac{1}{3}$.

$$s = 2i :$$

$$(((2Ai + B)(0) + (2Ci + D)(-3) = 4$$

so $C = 0$ (there is no imaginary part) and $-3D = 4$, so $D = -\frac{4}{3}$.

Similarly we can solve the part with the Heaviside function component and get $A = C = 0; B = -\frac{1}{3}; D = \frac{1}{3}$.

This allows us to rewrite the Laplace transform as

$$\frac{\frac{1}{3}}{s^2+1} + \frac{-\frac{4}{3}}{s^2+4} + e^{-2\pi s}\left(\frac{-\frac{1}{3}}{s^2+1} + \frac{\frac{1}{3}}{s^2+4}\right)$$

and the inverse Laplace transform of this is

$$\frac{1}{3}\sin(t) - \frac{2}{3}\sin(2t) + H(t-2\pi)(-\frac{1}{3}\sin(t) + \frac{1}{6}\sin(2t))$$

Remember that the Laplace transform of $\sin(2t)$ is $\frac{2}{s^2+4}$, so the inverse Laplace transform of $\frac{1}{s^2+4}$ is $\frac{1}{2}\sin(2t)$.

Without the Heaviside function, we have $y = \frac{1}{3}\sin(t) - \frac{2}{3}\sin(2t)$ for $t < 2\pi$ and $y = -\frac{1}{2}\sin(2t)$ for $t > 2\pi$ (I made a sign error in class, I think).

3.

$$y'' + y = h(t), y(0) = 0, y'(0) = 0$$

where $h(t) = \cos(2t)$ for $t < \frac{\pi}{2}$

and $h(t) = 0$ for $t \geq \frac{\pi}{2}$

Be careful in this problem – notice that π is only half of a period of the function $\cos(2t)$!

The function $h(t) = \cos(2t) - H(t - \frac{\pi}{2})\cos(2t)$.

The Laplace transform of $H(t-\frac{\pi}{2})\cos(2t)$ is $e^{-\frac{\pi}{2}s}$ times the Laplace transform of $\cos(2(t+\frac{\pi}{2}))$, which is $e^{-\frac{\pi}{2}s}$ times the Laplace transform of $-\cos(2t))$, or $\frac{-e^{-\frac{\pi}{2}s}}{s^2+4}$.

The equation is

$$s^2L\{y\}+L\{y\}=\frac{s+e^{-\frac{\pi}{2}s}}{(s^2+1)(s^2+4)}$$

The partial fraction decomposition setup is the same as in the previous problem.

For both parts, we need to solve the equation

$$(As+B)(s^2+4)+(Cs+D)(s^2+1)=s$$

to which the solutions are $A=\frac{1}{3}; B=0; C=-\frac{1}{3}; D=0$. I give the calculation in the style using complex numbers that I demonstrated above. It could of course also be done using simultaneous equations the way I did the last problem in class.

$$s=i:$$

$$(Ai+B)(3)+(Ci+D)(0)=i$$

so $3B=0$ (there is no real part) and $3A=1$, so $A=\frac{1}{3}$.

$$s=2i:$$

$$(((2Ai+B)(0)+(2Ci+D)(-3)=2i$$

so $D=0$ (there is no real part) and $-6C=2$, so $C=-\frac{1}{3}$.

This means that the Laplace transform of y can be rewritten as

$$\frac{\frac{1}{3}s}{s^2+1}+\frac{-\frac{1}{3}s}{s^2+4}+e^{-\frac{\pi}{2}s}\left(\frac{\frac{1}{3}s}{s^2+1}+\frac{-\frac{1}{3}s}{s^2+4}\right)$$

The inverse Laplace transform of this is

$$\frac{1}{3}\cos(t) - \frac{1}{3}\cos(2t) + H(t-\frac{\pi}{2})(\frac{1}{3}\cos(t-\frac{\pi}{2}) - \frac{1}{3}\cos(2(t-\frac{\pi}{2})))$$

which (applying trig identities to get rid of $(t-\frac{\pi}{2})$ in the Heaviside part) is equal to

$$\frac{1}{3}\cos(t) - \frac{1}{3}\cos(2t) + H(t-\frac{\pi}{2})(\frac{1}{3}\sin(t) + \frac{1}{3}\cos(2t))$$

so $y = \frac{1}{3}(\cos(t) - \cos(2t))$ for $t < \frac{\pi}{2}$ and $y = \frac{1}{3}(\cos(t) + \sin(t))$ for $t > \frac{\pi}{2}$.