Outline Proof of the Completeness Theorem with Questions

Randall Holmes
November 26, 2007

In this document I will outline a proof of the Completeness Theorem for first-order sequent calculus, with questions where you are asked to fill in details.

First, what are we trying to prove? We are trying to show that any statement which has to be true is in fact provable. It is equivalent to say that any statement which cannot be proved is in fact false under some interpretation. We generalize this to sequents: we claim that for any sequent $\Gamma \vdash \Delta$ which is not provable, we can construct an interpretation under which every statement in $\Gamma$ is true and every statement in $\Delta$ is false: so what we show is that every nonprovable sequent is invalid, which is equivalent to saying that every valid sequent is provable.

So we begin by assuming that $\Gamma \vdash \Delta$ is a sequent in the language of first order logic which cannot be proved.

The language of first-order logic which we consider at least contains negation, disjunction, the existential quantifier, predicates and variables. It can contain function symbols for building terms as well; this has no direct effect on the way the proof works.

We construct a sequence $\Gamma_i \vdash \Delta_i$ by recursion.

The assumption we make at stage $i$ (which you may verify holds at stage 0) is that $\Gamma_i \vdash \Delta_i$ is not provable in sequent calculus, each of the formulas $F_j$ for $j < i$ is either in $\Gamma_i$ or in $\Delta_i$, and $\Gamma_j \subseteq \Gamma_{j+1}$ for each $j < i$.

1. If $F_i$ is not of the form $(\exists x. \phi(x))$ and $\Gamma_i \vdash F_i, \Delta_i$ is provable, then we define $\Gamma_{i+1}$ as $\Gamma_i \cup \{F_i\}$ and $\Delta_{i+1}$ as $\Delta_i$. 


2. If $F_i$ is of the form $(\exists x. \phi(x))$ and $\Gamma_i \vdash F_i, \Delta_i$ is provable, then we let $y$ be the first variable not appearing in $\Gamma_i \vdash F_i, \Delta_i$, and we define $\Gamma_{i+1}$ as $\Gamma_i \cup \{F_i, \phi(y)\}$ and $\Delta_{i+1}$ as $\Delta_i$.

**Exercise 1:** Explain why

$$\Gamma_i \vdash F_i, \Delta_i$$

being provable (along with our inductive hypotheses about $\Gamma_i \vdash \Delta_i$) implies that

$$\Gamma_i, F_i \vdash \Delta_i$$

is not provable (this uses the cut rule).

**Exercise 2:** Explain why

$$\Gamma_i, (\exists x. \phi(x)) \vdash \Delta_i$$

not being provable implies

$$\Gamma_i, (\exists x. \phi(x)), \phi(y) \vdash \Delta_i$$

not provable when the variable $y$ does not appear in $\Gamma_i, (\exists x. \phi(x)) \vdash \Delta_i$. (this is direct from rules of sequent calculus).

3. If $\Gamma_i \vdash F_i, \Delta_i$ is not provable, we define $\Gamma_{i+1}$ as $\Gamma_i$ and $\Delta_{i+1}$ as $\Delta_i \cup \{F_i\}$.

Exercises 1 and 2 contain the work needed to show that $\Gamma_{i+1} \vdash \Delta_{i+1}$ is not provable; the other inductive hypotheses obviously remain true. So this construction can be carried out for all $n$.

Define $\Gamma_\infty$ as the union of all $\Gamma_n$’s and $\Delta_\infty$ as the union of all $\Delta_n$’s. Our intention is that the statements in $\Gamma_\infty$ will be the true statements in our interpretation and the statements in $\Delta_\infty$ will be the false statements.

**Exercise 3:** Explain why each formula of our language belongs to one of $\Gamma_\infty$ and $\Delta_\infty$ and not to both.

For simplicity, we consider a version of the proof in which equality is not in our language.

In this case, the elements of the domain of our model will be the terms of our language.

We interpret a unary predicate $P$ as holding of a term $t$ in the model iff the sentence $P(t)$ belongs to $\Gamma_\infty$. Similarly, we interpret a binary relation $R$
in our language as holding between terms $t$ and $u$ (considered as elements of the model) just in case $t R u \in \Gamma_{\infty}$, and similarly for predicates of higher arity in the language used.

To complete the proof, we need to be sure that the logical constructions (negation, disjunction, existential quantifier) have the correct behavior.

**Exercise 4:** Show that $\neg P \in \Gamma_{\infty} \iff P \notin \Gamma_{\infty}$.

**Exercise 5:** Show that $P \lor Q \in \Gamma_{\infty}$ if and only if either $P \in \Gamma_{\infty}$, $Q \in \Gamma_{\infty}$ or both.

**Exercise 6:** Show that $(\exists x. P(x))$ is in $\Gamma_{\infty}$ if and only if there is a term $t$ such that $P(t) \in \Gamma_{\infty}$.

Once we have established that atomic sentences have the intended truth values and logically complex sentences have the correct values indicated by their structure, we see that we have a model in which all the sentences in $\Gamma$ are true and all the sentences in $\Delta$ are false.

Moreover, we see that we have a finite or countable model in which these things are true. Any theory from whose axioms we cannot deduce a contradiction and which has a finite or countable language has a finite or countable model.

I simplified matters here by leaving out equality. There are two ways to fix this. If we introduce equality to our language, then we need to modify our model. It cannot be made up of all terms of the language, because there are likely to be distinct terms $t$ and $u$ with the same referent. There are two ways to handle this: model elements can be taken to be equivalence classes of terms under the relation $t \sim u$ defined by $t = u \in \Gamma_{\infty}$. The model element representing the referent of a term $t$ could also be taken to be the first term $T_i$ in an enumeration of the terms of the language such that $T_i \sim t$; in this way the model elements would still be terms rather than classes of terms but not all of them. I’m not going to ask you to fix up the proof with equality added.

If a language has only finitely many primitive predicates, then a predicate with all the logical properties of equality can be defined. The idea is to take the conjunction of all sentences $x =^P_i y$ of the form $(\forall z_1, \ldots, z_n.P(z_1, \ldots, z_{i-1}, x, z_{i+1}, \ldots, z_n) \iff P(z_1, \ldots, z_i, y, z_{i+1}, \ldots, z_n))$ (with the obvious modifications when $i = 1$ or $i = n_P$, over all primitive predicates $P$ of the language, where $n_P$ is the number of arguments taken by $P$): it can be proved that this will satisfy the
logical properties of equality in any theory with just those predicates. For example, if equality is the only primitive predicate (which is arguably true in full formalizations of mathematics!) then $(\forall z. z \in x \leftrightarrow z \in y) \land (\forall z. x \in z \leftrightarrow y \in z)$ is the defined equality relation. The defined equality will be represented by an equivalence relation on terms in the model constructed in our basic proof; this will not actually be equality, but it will have the right properties in relation to the primitive predicates of the model (the same statements will be true of “equal” terms). (This maneuver for introducing equality will be much more complicated in a language with function symbols, though it can be done in principle).