Review for Test III Makeup Opportunity, Math 107: extended Nov 27 with extra problems and some solutions; extended Nov. 29 with solutions to all problems – except solutions to curve sketching problems are handwritten and available at my office door

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The makeup will be given on Friday after the Thanksgiving break (November 30). You will have the opportunity to do questions which will replace certain questions on your Test III paper (you cannot make your Test III grade worse by doing the makeup). The makeup will not cover the entire test. You will be allowed to do a curve sketching problem (5 or 6), a word problem (7) and a problem replacing either 1 or 8 (linearization or L'Hôpital's Rule). If you do more than three problems on the makeup your best performance in each category will be used.

There are no makeups for the makeup: you need to be in class on November 30th to take advantage of this opportunity. There will be other opportunities to improve scores on selected problems on earlier tests built into the final exam.

This document contains (for the moment) review word problems. It would be a good idea to review basic sketching of rational functions for the problem 5 analogue. This you should be able to do with any precalculus book. There are some problems in the book similar to problem 6; you should look at these. Reviewing for linearizations (3.10) or L'Hôpital's Rule (4.6) you should be able to do with your book.

I might add problem 5 and 6 review problems to this document later (no promises); I will definitely add detailed solutions to the word problem examples, but definitely not for a few days. If you look at my main web page it will tell you when I have updated and how.

If you do these word problems after I post the solutions, please resist the temptation to look at the solutions before you have worked the problem and checked it yourself. I suggest using a grapher to check where maxima and minima of functions actually occur (sometimes the functions don't graph well of course). You can't do this on the exam but it is a good skill to develop.

1 The Word Problems

Solutions to the word problems now appear in the last section. Don't look until you have tried the problems yourself!

1. A sheet of cardboard is 5 feet by 8 feet. Squares of equal size are to cut out of the corners and the flaps folded up to create a box. How large should the squares be to make the volume of the box as large as possible?

   Include a verification that you have found at least a local maximum in your work (the easiest way to do this is to use the second derivative test).

2. A box with cardboard sides and a square metal top and bottom is to be made to hold Tinker Toys. The metal top and bottom cost ten times per square inch as much as the cardboard sides. The volume of the box is to be 2000 cubic inches. What dimensions should the box have to minimize the cost of the materials? (You do not need to know the price per square inch of cardboard or metal to solve the problem, but you are free to choose a price for the cardboard and set the price for the metal at ten times as much)

   Round your answer to two decimal places. Include a verification that you have found at least a local minimum in your work (the easiest way to do this is to use the second derivative test).

3. A girl sells lemonade on the steps of the State Capitol on a hot summer day. If she sells lemonade for 40 cents per cup, she will sell 500 cups in a day. For each cent she lowers the price, she will sell 20 more cups (and for each cent she raises it, she will sell 20 fewer cups). It costs her 5 cents per cup to prepare the lemonade. How many cups of lemonade should she sell at what price to maximize her profit?

   Hints: profit is revenue minus cost. It is easiest to set up this problem in terms of the number of cents that she lowers the price. Be careful that you state the answer that is actually being asked for! Give evidence that you actually have a maximum.

4. A painting 10 feet high hangs with its bottom 20 feet above the floor. A five foot tall woman is viewing the painting. How far from the painting should she stand to make her angle of view for the painting as large as possible? (Hint: draw a picture. The angle which is to be minimized can be expressed as the difference of two arc tangents. When you take derivatives of arc tangents everything becomes algebraic).

2 Additional Problems

Solutions to these will appear by Thursday. They now appear in the last section at the end of this document. Don't look at them until you have tried the problems yourself! Solutions to the curve sketching problems are hand written and must be picked up at my office door.

1. Find the linear function which approximates $y = \sqrt{x}$ near $x = 4$. Use this function to approximate $\sqrt{4.02}$ and $\sqrt{3.97}$.

2. One of these practice problems requires a single application of L'Hôpital's Rule; one requires two or more applications, and one requires none. Find each limit. Indicate briefly (arrows work) why the rule applies when it does apply and why it doesn't when it doesnt.

   (a) $\lim_{x\to 1} \frac{\ln(x)}{x^2-1}$

   (b) $\lim_{x\to 0} \frac{x}{e^x}$

   (c) $\lim_{x\to\infty} \frac{x^2}{e^{2x}}$

3. The function

$$f(x) = \frac{x^2 - 4}{(x-1)^2}$$

has first and second derivatives

$$f'(x) = \frac{-2(x-4)}{(x-1)^3}$$

$$f''(x) = \frac{4x - 22}{(x-1)^4}$$

Identify all horizontal and vertical asymptotes of the graph of f.

Make sign charts for f, f', and f''. Use these to indicate intervals where f is positive and where it is negative, where f is increasing and where f is decreasing, and where the graph of f is concave up and where it is concave down. Identify any local maxima or minima and points of inflection.

Sketch the graph of f using this information.

4. The following information is given you about a function g.

   Indicate any asymptotes of the graph of g, and intervals where g is increasing and where g is decreasing, and where the graph of g is concave up and where it is concave down. Identify any local maxima or minima and points of inflection.

   Then sketch the graph of g using this information.

   The domain of g, g', and g'' is the set of all reals except 1.

   $\lim_{x\to\infty} g(x) = \lim_{x\to-\infty} g(x) = 0$.

   $\lim_{x\to 1+} g(x) = \infty$

   $\lim_{x\to 1-} g(x) = -\infty$

   $g(.5) = 0$

   $g(x) > 0$ for $x \in (-\infty, .5)$ and $x \in (1, \infty)$

   $g(x) < 0$ for $x \in (.5, 1)$.

   $g'(-1) = 0; g(-1) = 2$

   $g'(x) > 0$ for $x \in (-\infty, -1)$

   $g'(x) < 0$ for $x \in (-1, 1)$ and $x \in (1, \infty)$

   $g''(-2) = 0$

   $g''(x) > 0$ for $x \in (-\infty, -2)$ and $x \in (1, \infty)$

   $g''(x) < 0$ for $x \in (-2, 1)$

3 Solutions to Word Problems

Don't look at these until after you try the problems yourself!

1. A sheet of cardboard is 5 feet by 8 feet. Squares of equal size are to cut out of the corners and the flaps folded up to create a box. How large should the squares be to make the volume of the box as large as possible?

   Include a verification that you have found at least a local maximum in your work (the easiest way to do this is to use the second derivative test).

   If you draw a picture (I'll draw this in class on request) you can see that if x is the length of one of the cuts, the box obtained has height x, length $8-2x$, width $5-2x$. So its volume is $x(5-2x)(8-2x) = 4x^3 - 26x^2 + 40x$ The derivative of this function is $12x^2 - 52x + 40$. I'd solve $12x^2 - 52x + 40 = 0$ for x using the quadratic formula: I can't factor that by inspection myself!

   $x = \frac{52 \pm \sqrt{52^2 - 4*12*40}}{24} = 1$ or $\frac{10}{3}$

   $x = \frac{10}{3}$ does not make sense, because x cannot be greater than 2.5 (half of 5). So the solution occurs when $x = 1$: the squares should be 1 foot by 1 foot.

   The second derivative is $24x - 52$, which is negative at $x = 1$, indicating a local maximum.

2. A box with cardboard sides and a square metal top and bottom is to be made to hold Tinker Toys. The metal top and bottom cost ten times per square inch as much as the cardboard sides. The volume of the box is to be 2000 cubic inches. What dimensions should the box have to minimize the cost of the materials? (You do not need to know the price per square inch of cardboard or metal to solve the problem, but you are free to choose a price for the cardboard and set the price for the metal at ten times as much)

   Round your answer to two decimal places. Include a verification that you have found at least a local minimum in your work (the easiest way to do this is to use the second derivative test).

   For simplicity, assume the cost of cardboard is one cent per square inch and the cost of the metal top or bottom is ten cents per square inch.

   The length and width of the base of the box are equal: represent this number by x. Represent the height of the box by h.

   We are given the condition $x^2h = 2000$.

   We are to minimize the quantity $10(2x^2)+1(4xh)$, the cost of materials for making the box. The $2x^2$ is the area of the metal top and bottom (which costs 10 per square unit) and the $4xh$ is the total area of the 4 cardboard sides.

   Before we can use calculus, we need to put this all in terms of one variable. Because $x^2h = 2000$, we can write $h = \frac{2000}{x^2}$, and write the quantity to be minimized as $10(2x^2) + 1(4x(\frac{2000}{x^2})$, which simplifies to $20x^2 + \frac{8000}{x}$. The derivative of this is $40x - \frac{8000}{x^2}$. Solving for critical points: $40x - \frac{8000}{x^2} = 0$, so $40x = \frac{8000}{x^2}$, so $40x^3 = 8000$, so $x^3 = 200$: the only critical point is at $x = \sqrt[3]{200} \sim 5.85$. $h = \frac{2000}{x^2} \sim 58.48$. It appears that the optimal thing is to make the box ten times as high as it is long or wide, which seems sensible.

   The second derivative is $\frac{16000}{x^3}$, which is positive at the critical point and in fact for all x which make sense in the problem (all positive values); so this is a minimum.

3. A girl sells lemonade on the steps of the State Capitol on a hot summer day. If she sells lemonade for 40 cents per cup, she will sell 500 cups in a day. For each cent she lowers the price, she will sell 20 more cups (and for each cent she raises it, she will sell 20 fewer cups). It costs her 5 cents per cup to prepare the lemonade. How many cups of lemonade should she sell at what price to maximize her profit?

   Hints: profit is revenue minus cost. It is easiest to set up this problem in terms of the number of cents that she lowers the price. Be careful that you state the answer that is actually being asked for! Give evidence that you actually have a maximum.

   Let x be the number of cents that she lowers the price (this is in practical terms the easiest way to set up the problem: it isn't the way an economist would do it, because they always want to put everything in terms of production level).

   Her revenue is the number of cups she sells times the price per cup. The price per cup is $40 - x$ (x is defined as the number of cents she lowers the price). The number of cups she sells is $500 + 20x$ (she sells twenty more cups for each cent the price is lowered). So her revenue is $(40 - x)(500 + 20x)$.

   Her cost is 5 cents per cup she makes: this is $5(500 + 2x)$.

   Her profit is then $(40 - x)(500 + 20x) - 5(500 + 20x) = (35 - x)(500 + 20x) = -20x^2 + 200x + 17500$. The derivative is $-40x + 200$, which is equal to 0 when $x = 5$: she should lower the price by 5 cents, and sell $500{+}(20)(5) = 600$ cups of lemonade at 35 cents each.

   The second derivative of the profit function is -4, which is negative everywhere so we see that we have a maximum.

4. A painting 10 feet high hangs with its bottom 20 feet above the floor. A five foot tall woman is viewing the painting. How far from the painting should she stand to make her angle of view for the painting as large as possible? (Hint: draw a picture. The angle which is to be minimized can be expressed as the difference of two arc tangents. When you take derivatives of arc tangents everything becomes algebraic).

   This is a more challenging problem.

   If you draw the picture correctly (I will do this in class) you will see that the angle from the top of the woman's head (which for simplicity we take to be eye level) to the bottom of the painting is $\arctan(\frac{15}{x})$ and to the top of the painting is $\arctan(\frac{25}{x})$, where x is her distance from the wall on which the painting is hanging.

   So the quantity to be maximized is $\arctan(\frac{25}{x}) - \arctan(\frac{15}{x})$.

   I'm leaving this as it is for now (this is more complicated than anything that will appear on the makeup; but do try computing the derivatives and see if you can solve for the critical point). Notice that when you take the derivative what you get will be purely algebraic: solving for zeroes of this will reduce to solving a quadratic, which my calculator tells me has solutions $\pm 5\sqrt{15}$; the negative solution isn't meaningful, and she should stand about 19.3 feet from the painting for the best view.

4 Additional Problems, With Solutions

Solutions to these will appear by Thursday. They now appear in the last section at the end of this document. Don't look at them until you have tried the problems yourself! The online version will not have pictures of graphs; I will post pictures of the graphs in these problems on my office door sometime this afternoon. For problem 5 you can of course try your graphing calculator if you have one, though this does not always give nice results (if major features of the graph happen not to be in the window you are looking at).

1. Find the linear function which approximates $y = \sqrt{x}$ near $x = 4$. Use this function to approximate $\sqrt{4.02}$ and $\sqrt{3.97}$.

   The linear function $L(x)$ which approximates $f(x)$ near $x = a$ is

   $$L(x) = f(a) + f'(a)(x - a)$$

   or in this case $L(x) = \sqrt{4} + \frac{1}{2\sqrt{4}}(x - 4) = 2 + \frac{1}{4}(x - 4)$.

   The approximations are $L(4.02) = 2 + \frac{1}{4}(.02) = 2.005$ and $L(3.97) = 2 + \frac{1}{4}(-.03) = 1.9925$

2. One of these practice problems requires a single application of L'Hôpital's Rule; one requires two or more applications, and one requires none. Find each limit. Indicate briefly (arrows work) why the rule applies when it does apply and why it doesn't when it doesnt.

   (a) $\lim_{x\to 1} \frac{\ln(x)}{x^2-1}$

   L'Hôpital's Rule applies, once. As x goes to 1, both the numerator and denominator go to 0, so $\lim_{x\to 1} \frac{\ln(x)}{x^2-1} = \lim_{x\to 1} \frac{\frac{1}{x}}{2x} = \frac{1}{2}$ (just plug in $x = 1$ in the second step!)

   (b) $\lim_{x\to 0} \frac{x}{e^x}$

   L'Hôpital's Rule does not apply. The numerator goes to 0 and the denominator to 1, so the limit is 0.

   (c) $\lim_{x\to\infty} \frac{x^2}{e^{2x}}$

   L'Hôpital's Rule applies, twice. $\lim_{x\to\infty} \frac{x^2}{e^{2x}} = \lim_{x\to\infty} \frac{2x}{2e^{2x}} = \lim_{x\to\infty} \frac{2}{4e^{2x}} = 0$ (in the first two steps, numerator and denominator go to infinity so the rule applies: in the last step the numerator goes to 2 and the denominator to infinity so the limit is 0).

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