Math 187 Test III

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This exam begins at 1:40 pm and ends at 2:35 pm. You may use a standard scientific calculator with no graphing or symbolic computation capability. Cell phones to be turned off and out of sight.
1. Counting problems. This problem has four parts. Each one involves making a choice of $k$ objects from $n$ alternatives. Classify each problem depending on whether one can make the same choice repeatedly and whether the order in which the choices are made matters. Answer each question, showing setup and a final numerical answer.

(a) A businessman has 10 suits. He needs to pack 3 of them to go on a trip. How many ways can he do this?

(b) A multiple choice test has 8 questions, each with 3 possible answers. In how many ways can the test be filled out?

(c) I want to order a dozen bagels: there are poppy seed, onion and sesame bagels available today. In how many different ways can I fill my order?

(d) A bag contains 26 Scrabble tiles, each with a different letter on it. I draw 3 tiles and arrange them on the tray in front of me. How many “words” can I form in this way?
2. Give the initial setup for a proof by contradiction of the statement “If $x$ and $y$ are prime numbers and $x + y$ is prime, then either $x = 2$ or $y = 2$”. You do not need to complete the proof, just write the beginning of the proof.
3. Prove by mathematical induction that the sum of the first \( n \) odd numbers is \( n^2 \): 
\[
1 + 3 + 5 + \ldots + (2n - 3) + (2n - 1) = n^2.
\]
As part of your setup, write this statement in summation (\( \Sigma \)) notation: you do not have to use this notation in your proof, though.

Be sure that the basis step, induction step, inductive hypothesis and any use of the inductive hypothesis in the proof are clearly identified. Make sure that you clearly distinguish between what you are assuming and what you are trying to prove.
4. Prove by mathematical induction that \( n^3 + 5n \) is divisible by 3 for each natural number \( n \).

Be sure that the basis step, induction step, inductive hypothesis and any use of the inductive hypothesis in the proof are clearly identified. Make sure that you clearly distinguish between what you are assuming and what you are trying to prove.
5. Compute the first six terms of the sequence defined by the equations

\[ a_0 = 1; a_1 = 3; a_{n+2} = 6a_{n+1} - 8a_n. \]

Derive a formula for \( a_n \) of the form \( a_n = c_1 A^n + c_2 B^n \). Show complete work for the derivation. Use the formula to compute \( a_9 \).
6. Functions

(a) Write the set of ordered pairs which implements the function from \{1, 2, 3, 4, 5\} to \{x \in \mathbb{N} \mid x \leq 25\} defined by \(f(x) = x^2\).

(b) How many functions are there from \{1, 2, 3\} to \{a, b\}?

How many of them are one-to-one? If there is one, give one such function in the form of a list of ordered pairs.

How many of them are onto \{a, b\}? If there is one, give one such function in the form of a list of ordered pairs (Hint: you do not have a formula for the number of onto functions: but you can in this case easily count how many functions are not onto).

(c) How many functions are there from \{a, b, c\} to \{1, 2, 3, 4\}?

How many of them are one-to-one? Give one such function in the form of a list of ordered pairs, if there are any.

How many of them are onto? If there is one, give one such function in the form of a list of ordered pairs.
7. Compositions

(a) Define $f$ as $\{(1, a), (2, a), (3, b), (4, c)\}$ and $g$ as $\{(a, 1), (b, 3), (c, 5)\}$. Give the domain and the image of each of these functions.

Compute $f \circ g$ (give it as a set of ordered pairs) or explain why it is undefined.

Compute $g \circ f$ (give it as a set of ordered pairs) or explain why it is undefined.
(b) Define $f(x) = \frac{1}{x}$ for each real number $x$ not equal to 0 and define $g(x)$ as $x + 1$ for each real number $x$.

If you were asked to compute $f \circ g$ and $g \circ f$ in Math 147, you could write formulas for both of them. Write the formulas.

In Math 187, we are more careful: one of these compositions does not exist! Which one? Why?

How could we modify the definition of one of the functions to make it so that the “bad” composition in the previous paragraph exists?
8. section 24

(a) Choose 20 points in a 4 inch by 4 inch square. Use the pigeonhole principle to argue that two of these points must be at a distance $\leq \sqrt{2}$ inches from each other. For full credit, your explanation must clearly reveal that you understand what the Pigeonhole Principle is (be sure to state the inequality that shows that the Pigeonhole Principle applies) and that you know what the relevant geometry facts are.
(b) Describe a bijection between the natural numbers and the integers. A formula is not necessary: it is sufficient to show enough of it in a diagram or list of ordered pairs that it is clear that you know what the pattern is.