Math 187 Sample Test II Questions

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1. Prove that for any natural numbers x, y, z, if $x < y$ then $xz < yz$, using the definition of $<$ and the axioms from chapter 4 (which would be given).

2. Prove that for any x, y, z natural numbers with $y > z$, $x(y - z) = xy - xz$, using the axioms and the definition of subtraction (which would be given).

3. Prove using mathematical induction that

$$(\Sigma_{i=1}^{n} i) = \frac{n(n+1)}{2}$$

4. The *Lucas numbers* are defined by the recursive definition

$$u_1 = 1; u_2 = 3; u_{n+2} = u_n + u_{n+1}$$

   Prove by induction that $u_{n+2} = f_n + 3f_{n+1}$ for each natural number n, where f_n refers to the nth Fibonacci number.

5. List all functions from the set $\{a, b, c\}$ to $\{1, 2\}$ (you may write them as sets of ordered pairs or illustrate them with arrow diagrams). Identify any of these functions which may be injections. Identify any of these functions which may be surjections.

6. Determine how many injections there are from $\{1, 2, 3\}$ into $\{a, b, c, d\}$; draw a tree diagram to illustrate your answer (you may choose not to draw the entire tree, but just enough to make it clear how many branches it has).

7. Look at the diagrams provided. Determine which of the following expressions involving compositions and inverses are meaningful and write down the ones which are meaningful as sets of ordered pairs.

   (a) fg

   (b) gf

   (c) h^{-1}

   (d) g^{-1}

   (e) $h^{-1}g$

8. Prove that if f and g are surjections, so is the composition gf. Prove the same result for injections.

9. Argue using the Pigeonhole Principle that from any four numbers a, b, c and d, one can choose two numbers whose difference is divisible by three.

10. Prove that the set of natural numbers all of whose digits in base ten are different is finite. Hint: this also uses the Pigeonhole Principle.

11. Define the function $f(n)$ recursively: $f(1) = 13$; if $f(n)$ is even,

$$f(n+1) = \frac{f(n)}{2}$$

; if $f(n)$ is odd,

$$f(n+1) = 3n + 1.$$

Compute enough values to show that f is neither an injection nor a surjection. You must explain correctly why f is not an injection and why f is not a surjection.

12. Prove that the relation on natural numbers defined by "xy is a perfect square" is an equivalence relation. Partition the numbers ≤ 20 into equivalence classes under this relation.