Counting Problems

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These problems are intended for use in studying counting principles. This is the version with the solutions.

1. (thanks to Dr. Grantham) Suppose you're trying to guess my password, and that you know I've been foolish enough to construct it using exactly the following 13 characters, once each: s,b,g (my initials, in lower case); 1,9,5,7 (the year I was born); and C,H,E,R,Y,L (my wife's name, in upper case).

   (a) If you knew nothing about the order in which I had used those characters, how many different possibilities would there be for my password?

   Solution: 13! (thirteen factorial). This is the number of ways to arrange thirteen distinct objects in order.

   (b) If you knew that I had kept the groups in the order listed (first my initials, then birth year, then wife's name), but may have scrambled the order *within* each group, then how many different possibilities would there be? [For example, bgs7915HLYREC would be one such possibility, but 5719gsbLRYCHE would not be, since the numbers precede the initials, and g57RECb91sHLY would not be, since the groups have been intertwined.]

Solution: (3!)(4!)(6!); choose from three factorial ways to order the lower case letters, then four factorial ways to order the numerals, then six factorial ways to order the upper-case letters.

(c) If you knew instead that I had kept the characters *within* each group in the relative order listed above, but that I might have "intertwined" the groups, then how many different possibilities would there be? [For example, CH1s9ERbg5Y7L would be one such possibility, but CH1s5ERbg9Y7L would not be, since the order of the 9 and 5 has been reversed.]

Solution: Here all we need to do is choose which positions will contain which kind of character (upper-case, lower-case, or numeral). The solution is

$$\binom{13}{3}\binom{10}{4}\binom{6}{6};$$

there are other ways to write this as well.

2. The reading list for a certain literature course consists of 7 novels by Ernest Hemingway and 9 by John Steinbeck. Each student must choose a set of 6 of those novels to read, with at least two by each author. How many different sets of 6 novels meet this criterion?

Solution: If six novels are chosen, with at least two by each author, the possible ways to choose are two from Hemingway, four from Steinbeck, three from each, or four from Hemingway, two from Steinbeck. The solution for each of these cases is easy to find; the cases are disjoint, so the results for the cases can be added:

$$\binom{7}{2}\binom{9}{4} + \binom{7}{3}\binom{9}{3} + \binom{7}{4}\binom{9}{2}.$$

3. How many 5-card poker hands contain at least 3 face cards? [Remember, there are 12 face cards total: J, Q, K of each of the 4 suits.

   Solution: We want the hands with exactly three face cards, the hands with exactly four face cards, and the hands with exactly five face cards. There are 12 face cards and 40 non-face cards:

$$\binom{12}{3}\binom{40}{2} + \binom{12}{4}\binom{40}{1} + \binom{12}{5}.$$

4. Suppose you want to buy a dozen bagels and that the varieties you like are onion, parmesan, poppy seed, sundried tomato, and sunflower.

   (a) How many different "distributions" among these five varieties are possible? [For example, one distribution is 2 onion, 3 parmesan, 1 poppy seed, 4 tomato, and 2 sunflower; another is 3 onion, 4 parmesan, 0 poppy seed, 5 tomato, and 0 sunflower.]

Solution: This is a "moving walls" problem. There are 12 bagels to choose and there are 5 varieties of bagel (I hope that you noticed the typo!); thus we will have 12 bagels and 4 walls in a representation of one of these distributions. The answer is

$$\binom{16}{4}.$$

(b) How many different "distributions" are possible if you decide to have at least one of each of those varieties?

Solution: The problem then reduces to distributing 7 bagels into five arieties. 7 bagels and four walls:

$$\binom{11}{4}.$$

5. How many integers between 1 and 1000 inclusive are divisible by at least one of 3, 5, or 7? [Notational hint: let A be the set of numbers divisible by 3, B the set of numbers divisible by 5, and C the set of numbers divisible by 7.]

Solution: There are 333 numbers divisible by 3, 200 numbers divisible by 5, and 142 numbers divisble by 7. Adding these numbers together counts the 66 numbers divisible by 15 twice, the 28 numbers divisible by 35 twice, the 47 numbers divisible by 21 twice, and the 9 numbers divisible by 105 three times.

The answer is 333 + 200 + 142(add up numbers divisible by 3, 5, 7) + 66 + 28 +

47(take away numbers added twice) + 9(add the numbers divisible by 105, which up to this point have been added three times then taken away three times).

6. Suppose we have a set of 54 baseball players to be partitioned into 6 teams of 9 players each.

   (a) How many different ways are there to do this, if we don't care about the order of the teams, and if the players are not "specialized", so that *any* set of 9 players can comprise a team? Explain your counting method clearly.

   Solution:

$$\frac{\binom{54}{9}\binom{45}{9}\binom{36}{9}\binom{27}{9}\binom{18}{9}\binom{9}{9}}{6!}$$

   One has to divide by six factorial because the order in which the teams are chosen does not matter.

   (b) Now suppose that the players are specialized to the following extent: there are 18 outfielders, 24 infielders, 6 pitchers and 6 catchers. Now how many ways are there to partition the players into teams, with each team consisting of 3 outfielders, 4 infielders, a pitcher and a catcher? Again, explain your counting method clearly.

   Solution:

$$\binom{18}{3}\binom{24}{4}\binom{6}{1}\binom{6}{1}$$

$$\binom{15}{3}\binom{20}{4}\binom{5}{1}\binom{5}{1}$$

$$\binom{12}{3}\binom{16}{4}\binom{4}{1}\binom{4}{1}$$

$$\binom{9}{3}\binom{12}{4}\binom{3}{1}\binom{3}{1}$$

$$\binom{6}{3}\binom{8}{4}\binom{2}{1}\binom{2}{1}$$

describes the sequence of choices to be made (the choices for the last team, where there is only one choice at each step, are left out); this product needs to be divided by six factorial for the same reasons as in the previous part.

7. Suppose that license plates consist of three letters (all upper-case) followed by three digits.

   (a) How many possible license plates are there?

   Solution:

   $$(26^3)(10^3)$$

   (b) How many license plates are there in which exactly one "A" appears?

Solution:

$$(25^2)(3)(10^3):$$

3 counts the number of places to put the A and 25^2 counts the choices of letters other than A to put in the other two places.

(c) How many license plates are there in which at least one zero appears?

Solution: There are

$$(26^3)(9^3)$$

plates in which no zero appears: the solution is

$$(26^3)(10^3) - (26^3)(9^3).$$

(d) How many license plates are there in which exactly one "A" *or* exactly one "8" (or both) appears?

Solution: There are

$$(25^2)(3)(10^3)$$

plates in which exactly one A appears and

$$(26^3)(3)(9^2)$$

plates in which exactly one 8 appears, and

$$(25^2)(3)(3)(9^2)$$

plates with exactly one A and exactly one 8: the solution is

$$(25^2)(3)(10^3) + (26^3)(3)(9^2) - (25^2)(3)(3)(9^2)$$

(the subtraction is there because the sum counts the one with both one A and one 8 twice).

(e) How many possible license plates would there be if one required that there be three digits and three uppercase letters, allowing repetitions, but allowed digits and letters to appear in any order (for example, A22Z3A would be an allowed plate).

Solution:

$$\binom{6}{3}(26^3)(10^3)$$

; the additional factor can be understood as choosing the 3 positions out of 6 where letters will be placed.

8. There are 36 kids (18 boys and 18 girls) who want to play Little League.

(a) How many ways are there to divide the Little Leaguers into four teams of nine players each? All that matters is the way the players are divided among the teams; the teams are not in any special order.

Solution:

$$\frac{\binom{36}{9}\binom{27}{9}\binom{18}{9}\binom{9}{9}}{4!}$$

The four factorial is there because the teams could be chosen in any order.

(b) How many ways are there to select a "gender-balanced" team (i.e., 5 boys and 4 girls or 4 boys and 5 girls) from the pool of 36 players?

Solution:

$$\binom{18}{5}\binom{18}{4} + \binom{18}{4}\binom{18}{5}$$

(c) How many ways are there to field four gender-balanced teams for the Little League season? Hint: first explain why there must be exactly two 5 boy/4 girl and exactly two 4 boy/5 girl teams. Then solve the problem. As in the first part, there is no special order on the teams, but you need to remember that there are two different kinds of teams.

Solution: If there were three 5 boy teams, there would be too few boys left for the fourth team; similarly, there cannot be three 5 girl teams, so there must be two of each.

$$\binom{18}{5}\binom{18}{4}\binom{13}{5}\binom{14}{4}\binom{8}{4}\binom{10}{5}\binom{4}{4}\binom{5}{5}$$

describes the number of ways to choose the teams (choose boys, then girls for each team; choose the two teams with 5 boys first). This overcounts by a factor of (2!)(2!), since the two 5-boy teams and the two five-girl teams can each be chosen in two orders, so the answer is the long product above divided by 4.

9. Suppose that there are 25 professors in the BSU math department, of whom 7 are statisticians and 12 are computer scientists.

   What is the largest number which could be the number of professors who are neither statisticians nor computer scientists? What is the smallest number?

   Solution: The largest number is 13 (if all statisticians are computer scientists) ; the smallest is 6 (if no statisticians are computer scientists).

   Draw a Venn diagram illustrating each of the two extreme situations. Be sure to indicate in each of the two diagrams the number of statisticians, the number of computer scientists, the number of professors who fall in the intersection of the two categories, and the number who fall in neither category.

   Solution: I'll leave this for you.

5. Suppose that thirty men and twenty women are on jury duty. If a twelve member jury is formed at random

a. How many possible juries are there?

Solution:

$$\binom{50}{12}$$

b. How many possible juries are there with six men and six women?

$$\binom{30}{6}\binom{20}{6}$$

10. Suppose that a fair six-sided die is rolled four times. How many different outcomes are possible? How many of the different outcomes involve rolling exactly four twos?

Solution: 6^4 possible outcomes. It looks as if there is a typo here; of course only one alternative involves rolling exactly four twos! Suppose that the die is rolled 7 times instead; then there would be 6^7 outcomes, of which $\binom{7}{4} 5^3$ would involve rolling exactly four twos (choose four of the seven rolls on which to roll 2's, then choose from the 5 other numbers for the 3 other positions).

11. Suppose that all students at a certain technical college must take either biology, physics, or chemistry, or some combination of these courses. If there are 140 students in the class of 1992, and 80 took physics, of whom 40 took *only* physics, 60 took biology, of whom 20 took *only* biology, and 50 took chemistry, of whom 30 took *only* chemistry, and 10 students took both physics and chemistry, how many students took

all three courses?

Solution: 50 took chemistry, 30 took only chemistry and 10 took physics and chemistry, so 10 took chemistry, biology, and not physics. 60 took biology, of whom 20 took only biology, so 30 students took physics and biology (these may have taken chemistry as well). If we count the 30+40+20 who took only one subject, plus the 10 who took chemistry, biology and not physics, plus the 10 who took physics and chemistry, plus the 30 who took physics and biology, we have counted everyone just once, except those who took all three subjects, who have been counted twice. These numbers add to 140, which is the total number of strudents; thus there are no students who took all three subjects, since we have not overcounted anyone!

12. Of a group of students taking history, math, and chemistry, 9 got A's in history, of whom 4 had this as their only A, while 11 got A's in math, of whom 7 had this as their only A, and 12 got A's in chemistry, with 9 of these having this as their only A. 6 students didn't get any A's at all. 3 got A's in both history and math, 1 got A's in math and chemistry, and 2 got A grades in both history and chemistry. How many students got an A in all three subjects? How many students were there in the whole group? Draw a Venn diagram...

    Solution: To see how many got A's in all three subjects, look just at the students with A's in math. 7 had A's only in math, 3 had A's in history and math, and 1 got an A in both math and chemistry. The sum of these three numbers will count each person with an A in math alone or in math and one other subject once, and count each person with three A's twice. 7+3+1 = 11, the total number of A students in math; no one is overcounted, so no one has three A's.

The total number of students is now easy to compute: 4+7+9 (A's in one subject) + 3+1+2 (A's in two subjects; these sets are disjoint because there are no triple majors) + 6 (no A's at all).