Additional M187 Notes for Feb 10

Dr. Holmes

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In these notes, I'll discuss aspects of the definition of function that are not discussed in sections 5.1 and 5.2. These have mostly to do with the representations of functions as ordered pairs.

For any objects a and b, we suppose that there is an ordered pair (a, b). Two ordered pairs (a, b) and $c, d)$ are equal just in case $a = c$ and $b = d$. In particular, $(a, b) \neq (b, a)$ unless $a = b$: the pairs $(2, 4)$ and $(4, 2)$ are different, unlike the sets $\{2, 4\}$ and $\{4, 2\}$. This means that we cannot understand (a, b) as being the same thing as $\{a, b\}$. It is possible to define ordered pairs as being sets: the usual definition is $(a, b) = \{\{a\}, \{a, b\}\}$. But we will not use such a definition officially.

We will refer to the two "parts" a and b of a pair (a, b) as the "first projection" and the "second projection" of (a, b).

If A and B are sets, there is a set $A \times B$, called the cartesian product of A and B, having as its elements all the ordered pairs (a, b) with $a \in A$ and $b \in B$. For example, if $A = \{1, 2, 3\}$ and $B = \{a, b\}$ then $A \times B = \{(1, a), (2, a), (3, a), (1, b), (2, b), (3, b)\}$. Two things to note: there is a relation with multiplication: if A and B are finite sets with m and n elements, respectively, then $A \times B$ has mn elements. Note that though commutativity of multiplication tells us that $B \times A$ has the same number of elements, it is not the same set. In our example, $B \times A = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$: both of the cartesian products have $(2)(3) = 6$ elements, but they are not the same set (because reversing the order of an ordered pair makes a different pair). We won't make a lot of use of cartesian products at this point.

The book defines a function as a correspondence between a set A and a set B in which each element of A corresponds to exactly one element of B. We are more explicit: we define the function as the set of ordered pairs (a, b) such that a and b correspond.

Officially, a function from A to B is a set of ordered pairs f such that for all $(x,y) \in f$, $x \in A$ and $y \in B$, and for any x, y, z, $(x,y) \in f$ and $(x,z) \in f$ implies $y = z$: this expresses the idea that each element of A corresponds to no more than one element of B. Finally, for any $x \in A$ there is $y \in B$ such that $(x,y) \in f$; in fact, there is only one such y and we call it $f(x)$. We call A the domain of f and B the range of f.

A function is a *surjection* from A to B if for every element y of B there is at least one element x of A such that $y = f(x)$ (equivalently, $(x,y) \in f$). This property is also called "onto".

A function f is an *injection* from A to B iff for all $x, y \in A$, $f(x) = f(y) \rightarrow x = y$; this property is also called "one-to-one".

A function which is both an injection and a surjection from A to B is called a *bijection*. Bijections have inverse functions, as we shall see. If there is a bijection from a finite set A to a finite set B, it is easy to see that the sets A and B are the same size.

Functions may be defined using formulas (in the familiar format of examples like $f(x) = 2x + 1$). Functions on the natural numbers can also be defined by recursion. If a is a function with domain the set of natural numbers $\mathcal{N}$, we frequently write a_n instead of a_n; a function with domain the natural numbers is often called a *sequence*.

Some questions I asked at the end of class:

How many functions are there from $\{1,2\}$ to $\{1,2\}$? The class came up with the answer 4.

How many functions are there from $\{1,2,3\}$ to $\{1,2\}$? The class came up with the answer $2^3 = 8$.

How many functions are there from $\{1,2\}$ to $\{1,2,3\}$? The class answer was $3^2 = 9$.

How many functions are there from $\{1,2,3\}$ to $\{1,2,3\}$? The class answers was $3^3 = 27$.

In general, there are n^m functions from a set of size m to a set of size n. This can be seen by thinking of the fact that we need to make m choices (one for each element of the domain) with n different choices available at each step.

The last thing I asked in class was how many *injections* there are from $\{1,2,3\}$ to $\{1,2,3\}$? Is there a general way to answer this kind of question?