Test IV Review, Math 187

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section 10.6: Be able to write a function from $\{1, \ldots, n\}$ in permutation notation.

Sample problem: cards labelled 1,2,3,4,5,6 are cut and shuffled to get the order 5,2,6,3,1,4. Represent this function on $\{1, 2, 3, 4, 5, 6\}$ using permutation notation. How many times would you need to repeat this cut and shuffle combination to return the deck to its original state?

Solution: The cycle notation for this is (15)(2)(364) and it will take six cut/shuffles to restore the deck to its original state.

Be able to compute the composition of two functions in permutation notation or the inverse of a function written in permutation notation.

Sample problem: Compute the composition of the permutations (12)(34) and (14)(32) of $\{1, 2, 3, 4\}$ and write the result in permutation notation.

Solution: the composition is (13)(24).

The map (14)(32) sends 1 to 4; the map (12)(34) sends 4 to 3. So the composition sends 1 to 3.

The map (14)(32) sends 2 to 3; the map (12)(34) sends 3 to 4. So the composition sends 2 to 4.

The map (14)(32) sends 3 to 2; the map (12)(34) sends 2 to 1. So the composition sends 3 to 1.

The map (14)(32) sends 4 to 1; the map (12)(34) sends 1 to 2. So the composition sends 4 to 2.

The composition sends 1 to 3 and 3 back to 1 (thus (13) is a cycle in it). It sends 2 to 4 and 4 back to 2 (so (24) is a cycle in it). So the result is (13)(24).

Be able to tell how many times a function written in permutation notation needs to be applied to return to the identity. (this is found in the first sample problem).

sections 11.1, 11.2, 11.3: Be able to compute $\binom{a}{b}$ using multiplication and division (not just built-in statistics or combination functions of your calculator). You might also need to be able to compute them using Pascal's triangle.

Sample problem: Compute $\binom{5}{3}$ using addition, subtraction, multiplication and/or division. Compute it using Pascal's triangle.

Solution:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Here are five rows of the triangle: from the fifth row we read $\binom{5}{0} = 1$; $\binom{5}{1} = 5$; $\binom{5}{2} = 10$; $\binom{5}{3} = 10$.

Be able to apply binomial numbers to solve unordered selection problems with or without repetition.

Sample problem: How many subcommittees with three members can be formed from a committee with seven members?

solution: $\binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$

Sample problem: A bag contains at least 5 of each of the following colors of M&M's: red, green, brown, and orange. We draw 5 M&M's from the bag. How many possible handfuls of M&M's can I get (all I care about is how many M&M's of each color I get)?

solution: This is unordered choice with repetition, with $n = 4, r = 5$:
$\binom{5+4-1}{5} = \binom{8}{5} = \frac{8\cdot7\cdot6\cdot5\cdot4}{5\cdot4\cdot3\cdot2\cdot1} = 56$

Think of writing 0's for M&M's and 1's for changes of color: there will be 5 0's and 3 1's, so $\binom{8}{5}$ or $\binom{8}{3}$ will work.

In both sample problems, write your result as a binomial number, then compute its value using addition, subtraction, multiplication and/or division.

Be able to use binomial numbers (Pascal's triangle is a convenient way to compute them) to expand an expression of the form $(x + y)^n$.

Sample problem: Expand $(x+y)^8$, $(x-y)^8$, $(x+2y)^8$. Write the expansions in a form using binomial numbers, then evaluate the binomial numbers and write the expansions with evaluated coefficients.

Here's that triangle again:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

$(x+y)^8 = \binom{8}{0}x^8+\binom{8}{1}x^7y+\binom{8}{2}x^6y^2+\binom{8}{3}x^5y^3+\binom{8}{4}x^4y^4+\binom{8}{5}x^3y^5+\binom{8}{6}x^2y^6+\binom{8}{7}x^1y^7+\binom{8}{8}y^8 = x^8+8x^7y+28x^6y^2+56x^5y^3+70x^4y^4+56x^3y^5+28x^2y^6+8xy^7+y^8$

$(x-y)^8 = \binom{8}{0}x^8-\binom{8}{1}x^7y+\binom{8}{2}x^6y^2-\binom{8}{3}x^5y^3+\binom{8}{4}x^4y^4-\binom{8}{5}x^3y^5+\binom{8}{6}x^2y^6-\binom{8}{7}x^1y^7+\binom{8}{8}y^8 = x^8-8x^7y+28x^6y^2-56x^5y^3+70x^4y^4-56x^3y^5+28x^2y^6-8xy^7+y^8$

$(x+2y)^8 = \binom{8}{0} x^8 + \binom{8}{1} x^7(2y) + \binom{8}{2} x^6(2y)^2 + \binom{8}{3} x^5(2y)^3 +$
$\binom{8}{4} x^4(2y)^4 + \binom{8}{5} x^3(2y)^5 + \binom{8}{6} x^2(2y)^6 + \binom{8}{7} x^1(2y)^7 + \binom{8}{8} (2y)^8 =$
$x^8 + 16x^7y + 112x^6y^2 + 448x^5y^3 + 1120x^4y^4 + 1792x^3y^5 + 1792x^2y^6 +$
$1024xy^7 + 256y^8$

section 11.4: Solve Venn diagram problems using the sieve principle. Try it on the problems on the counting sheet (the solutions given there to not use this principle, but it can be used to solve that kind of problem).

A sieve principle problem may require you to solve for something unexpected:

Sample problem: A group of science students are taking physics, chemistry and/or biology. Each of the 50 students is taking one of these subjects. 40 take chemistry. 20 take biology. 10 take biology and physics 12 take chemistry and physics. 15 take chemistry and biology. 5 brave students take all three courses. How many students are taking physics? Solve this problem by setting up the equation for the total number of students using the sieve method and solving for the unknown number of physics students.

In a problem of this kind the setup using the sieve principle would be required for full credit.

Solution:

Let x stand for the number of students who take physics.

The setup is

$50 = (40+20+x)-(10+12+15)+5 = 60+x-37+5 = 60+x-32 = 28+x$ so $x = 50 - 28 = 22$. There are 22 students who take physics.

Warning about graph theory examples: Most sample problems in graph theory will involve hand drawings and so will not appear in the web version of this document!

section 15.1, 15.2: Be able to draw a graph if it is given using vertex and edge sets or using an adjacency table.

If two isomorphic graphs are presented, be able to write down an isomorphism between them as a function from the vertex set of one graph to the vertex set of the other.

If two graphs are not isomorphic, be able to present some explanation as to why they can't be (information about degrees of vertices is often used).

section 15.3: Understand what is meant by degree of a vertex in a graph.

Understand what is meant by a regular graph with a certain degree.

Be aware of the theorem that the number of vertices of odd degree in a graph must be even. It might even be a good idea to know how to prove this (the proof is easy).

sample problem: Draw a regular graph of degree 3 with five vertices, or explain why there can't be one.

solution: There can't be one, because we can't have 5 vertices of degree 3, since the number of odd vertices must be even.

sample problem: How many non-isomorphic degree 2 graphs with nine vertices are there? Hint: they are not all connected.

solution A degree 2 graph is a cycle. We can have a cycle with nine vertices, or a cycle with three and a cycle with six, or a cycle with four and a cycle with five, or three cycles with three each. There are 4 possibilities.

If you are given a list of degrees for vertices in a graph, be ready to draw a graph with vertices of those degrees or explain why it is impossible to draw one. The rule about even numbers of odd degree vertices might be used, along with common sense counting of vertices to be connected.

You might be asked to draw more than one nonisomorphic graph satisfying some conditions on number and degree of vertices. In such a question, I might ask you to explain why the graphs you present as nonisomorphic are in fact not isomorphic.

section 15.4: Paths and cycles.

Understand what it means for a graph to be connected and be able to identify the components of a graph.

Understand the definition of a cycle. Be able to identify 3-cycles or 4-cycles (or n-cycles in general) in graphs.

Understand the definition of an Eulerian walk and the conditions under which an Eulerian walk exists. If I present you with a graph, be ready to describe an Eulerian walk on the graph (by listing the vertices visited in order) or explain why there is no Eulerian walk on the graph.

Understand the definition of a Hamiltonian cycle and understand the statement of Oré's theorem (Oré's theorem was discussed in class, but is not in the book). The correct statement of the theorem is that any graph with n vertices with the property that any two non-adjacent vertices have degrees adding to at least n has a Hamiltonian cycle.

You should be ready if I present a graph and ask whether it has a Hamiltonian cycle to either present a Hamiltonian cycle (by listing vertices visited in order) or explain why there isn't one. Be aware that a graph may have a Hamiltonian cycle without satisfying the conditions of Oré's theorem! I might present you with a graph, tell you that it doesn't have a Hamiltonian cycle, then ask you how the conditions of Oré's theorem are violated.

It would be a good idea to remember the theorem that a bipartite graph with an odd cycle does not have a Hamiltonian cycle; I might present a graph which doesn't have a Hamiltonian cycle because of this theorem.

section 15.5: Be aware of the definition of a tree. Be aware of the fact that there is only one path from any given vertex to any other vertex in a tree.

Be aware of the theorem that the number of edges in a tree is one less than the number of vertices. I might ask you to draw a tree with a given number of edges and a given number of vertices; the theorem might enable you to say that such a problem has no solution. I might also ask you to draw more than one nonisomorphic tree satisfying some conditions.

section 15.6: Understand what the rules are for coloring a graph (the two vertices in any edge must have different colors).

Be able to tell if a graph is bipartite (can be colored with two colors). Understand that odd cycles prevent 2-colorings. Understand that a complete graph with k vertices requires $k+1$ colors.

Be able to determine the chromatic number of a graph if asked (what the minimum number of colors is that is required to color it). If asked to compute the chromatic number, give a coloring with that number of colors and explain why no smaller number of colors will do.

section 16.1: If I draw a tree, be able to redraw it using any chosen vertex as the root, organizing all vertices into levels and identifying the root, internal vertices and leaves.

Be aware of theorem 16.1: an m-ary tree with l leaves has height greater than or equal to $\log_m(l)$ (because an complete m-ary tree of height h has m^h leaves).

1 Solutions to Hand-Drawn Examples:

1.1 15.1-2:

The isomorphism sends a to 2, b to 0, c to 1 and d to 3. One could also send c to 3 and d to 1. (or of course we could go in the other direction: 2 to a, etc.)

1.2 15.3:

a is impossible: each of the three degree 4 vertices must connect to all 3 other vertices, so it is impossible for the first vertex to have degree just 1.

b is impossible: it has 3 (an odd number) odd vertices.

For part c I give an adjacency table:
A is connected to B,C,D,E
B is connected to A,C,F
C is connected to A,B
D is connected to A,E
E is connected to A,D,G
F is connected to B
G is connected to E

There may be other nonisomorphic solutions.

To get a graph with 8 vertices, degree 3, without 3-cycles, connect 1,2,3,4,5,6,7,8,1 in a cycle, then connect 1 and 5, 2 and 6, 3 and 7, 4 and 8 (connect pairs of opposite vertices).

To get a graph with 8 vertices, degree 3, with many 3-cycles, connect 1,2,3,4,5,6,7,8,1 in a cycle, then connect 1 with 3, 2 with 4, 5 with 7, 6 with 8.

You cannot draw a tree with 10 edges and 12 vertices, because a tree will always have one fewer edges than vertices.

1.3 15.4

The "star of David" is not connected: its components are the graphs with vertices 0,2,4 and 1,3,5.

The graph on $\mathcal{N}_{30}$ is also disconnected: the numbers divisible by 3 will make up one component, those of the form $3n + 1$ will make another, and those of the form $3n + 2$ will make another.

Eulerian walks: the graph on the left doesn't have one, since it has six odd vertices (we can only have two vertices of odd degree and have an E walk). The graph on the right has two vertices of odd degree, 3 and 6. This means we have to start at 3 or 6 and end at the other. 3,2,1,0,6,7,5,3,4,1,5,4,6 is a sample Eulerian walk (you might get a different one).

1.4 Hamiltonian Cycles

For the top graph, the conditions of Ore's theorem are not satisfied (each vertex is degree 3 and there are seven vertices: the sum of the degrees of any two vertices (and so of any two non-adjacent vertices) is 6 which is less than 7.

But there is a Hamiltonian cycle: one is 0,1,2,5,6,4,3,0.

For the graph below to the left, there are 6 vertices and the sum of the degrees of any two vertices is at least 6, so Ore's theorem is satisfied. This means that there must be a Hamiltonian cycle: an example is a,b,e,c,f,d,a.

The graph below and to the right is bipartite and has an odd number of vertices. Color 0,1,2 green and color 3,4,5,6 red, for example. This means that there cannot be a Hamiltonian cycle. Notice that the sum of the degrees of 5 and 6 (for example) is 6, which is less than the number of vertices, so we have no problem with Ore's theorem.

1.5 Chromatic Number examples

The complete graph of degree 5 needs 5 colors.

The graph at top right needs 3 colors: it can't be colored with 2 because alternating colors on the outer cycle causes opposite vertices (which are connected!) to be assigned the same color. Alternating red, green and blue on the outer cycle works.

The graph at the bottom can be colored with 2 colors (it is bipartite). Color 1,2,4,6,7 red and color 0,3,5 green.

1.6 Tree Rearrangements

first arrangement:

e is the root; b,g,f are level 1; a,c,d,h,i,j are level 2. The height of the tree is 2. a,c,d,f,h,i,j are leaves, e is the root, b,g,f are internal vertices.

second arrangement:

b is the root; a,c,d,e are level 1; f,g are level 2; h,i,j are level 3. The tree has height 3. a,c,d,f,h,i,j are leaves, b is the root, e,f,g are internal vertices.

Notice that the height of the tree depends on the choice of root. In these two examples the sets of leaves are the same. Can the same tree have different leaves depending on the choice of root? (Hint: the answer is Yes; how do you do this?)