Math 301 Test II

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This exam will begin at 9:40 am and end at 10:35 am. You may use a fancy calculator: be sure to follow directions as to what you are expected to show on your paper in particular questions. As a rule, if you are allowed to use a calculator on a question, you should show me what you did with the calculator. Cell phones must be turned off and out of sight.
1. Set up the two by two matrix representing the linear transformation
“rotate the plane through forty-five degrees counterclockwise around
the origin”.

Point out where in this matrix you can find the images of the standard
basis vectors of $\mathbb{R}^2$.

Set up the three by three matrix representing the same transformation
in homogeneous coordinates (section 2.7).

Set up the product of three-by-three matrices which represents the
transformation of the plane “rotate the plane through forty-five degrees
counterclockwise around the point (3,4)” in homogeneous coordinates.
Hint: you need to translate the point (3,4) to the origin, then rotate,
then translate it back. You do not need to evaluate this product of
matrices, just set it up.
2. Carry out each matrix multiplication or explain why it is undefined. Your explanation in the latter case should mention the domain and codomain (range) of the linear transformations represented by the matrices and use the fact that matrix multiplication corresponds to composition of functions. Show complete work by hand in this part (show the additions and multiplications you do in the parts where there is a product to compute).

(a) 
\[
\begin{bmatrix}
1 & 2 \\
4 & -1
\end{bmatrix}
\cdot
\begin{bmatrix}
3 & 1 \\
-2 & -5
\end{bmatrix}
\]

(b) 
\[
\begin{bmatrix}
1 & 2 \\
4 & -1
\end{bmatrix}
\cdot
\begin{bmatrix}
-1 & 0 & 3 \\
-2 & -1 & 4
\end{bmatrix}
\]

(c) 
\[
\begin{bmatrix}
-1 & 0 & 3 \\
-2 & -1 & 4
\end{bmatrix}
\cdot
\begin{bmatrix}
1 & 2 \\
4 & -1
\end{bmatrix}
\]
3. Find the inverse of the given matrix by row reduction: set up the appropriate larger matrix (write it down), compute its rref (for this you may use your calculator), and then write down the inverse matrix.

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}
\]

In this problem there is an inverse matrix; explain what features of a calculation of this kind would tell us that the inverse did not exist.
4. An equation involving partitioned matrices is shown. Find the matrices $X, Y, Z$ in terms of the given matrices $A, B, C$. Make sure you remember which algebraic rules do not work with matrices, and don't use them! You may assume that all matrices represented by capital letters are square and invertible.

\[
\begin{bmatrix}
X & 0 \\
Y & Z
\end{bmatrix} \cdot \begin{bmatrix}
A & 0 \\
B & C
\end{bmatrix} = \begin{bmatrix}
I & 0 \\
0 & I
\end{bmatrix}
\]
5. Find an $LU$ factorization for the given matrix. You need to show the construction step by step, indicating which row operation you apply at each stage and which column you add to $L$ at each stage. Remember that you cannot use scaling row operations or row interchanges.

\[
\begin{bmatrix}
1 & 2 & 1 & -3 \\
3 & 7 & 3 & -11 \\
-1 & 0 & -1 & 0 \\
\end{bmatrix}
\]
6. A matrix $A$ is given along with its row reduced echelon form. Give a basis for the column space of $A$ and a basis for the null space of $A$. Your answer should make it clear that you know what kind of mathematical object a basis is, and also should show me clearly that you understand the method for finding these bases.

\[
\begin{bmatrix}
1 & 2 & 1 & 5 \\
2 & 4 & 3 & 13 \\
-4 & -8 & 5 & 7
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 2 & 0 & 2 \\
0 & 0 & 1 & 3 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

For a couple of points, find a vector of the appropriate dimension to be in the column space of $A$ which is not in fact in the column space, and show a calculation which verifies this.
7. Prove that for any matrix \( A \), if the equation \( Ax = 0 \) has only the trivial solution, then the linear transformation sending each vector \( x \) to \( Ax \) is one-to-one. Do not attempt to apply the Invertible Matrix Theorem or any other “big” theorem.