Math 301 Test III

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The test will last from 9:40 am to 10:35 am. You may use your fancy calculator: as usual you must tell me what matrices you compute rrefs of and what the rrefs are. Books, notes and neighbors to remain firmly closed. Cell phones to be off and out of sight.
1. Find the area of the parallelogram with corners (1,2), (3,3), (2,5), and (4,6), by setting up and evaluating a suitable determinant. Show all work.
2. Determine the solution of the system of equations

\[
\begin{align*}
    x + y + z &= 3 \\
    x + 2y + z &= 4 \\
    x + y + 3z &= 5
\end{align*}
\]

using Cramer’s rule. Set up all calculations on your paper. You may evaluate determinants using your calculator.
3. Determine for which values of the constant $a$ the system of equations

$$ax_1 - 3x_2 = 0$$
$$x_1 + (a + 2)x_2 = 0$$

has nontrivial solutions. Use determinants to do this, and explain.
4. Evaluate the determinant by hand and intelligently, choosing rows or columns with few nonzero elements and using row or column operations to reduce the amount of work you have to do. Do not use the diagonal trick for evaluating three by three matrices; you may evaluate two by two matrices using the formula.

$$\begin{vmatrix}
1 & 1 & -1 & 4 \\
0 & 1 & 0 & 1 \\
2 & 1 & 1 & 17 \\
0 & -1 & 1 & 2
\end{vmatrix}$$
5. Explain briefly why the dimension of the null space and column space of a matrix $A$ must add up to the number of columns in $A$.

Find bases for the null space of $A$ and the column space of $A$ where $A$ is the matrix

\[
\begin{bmatrix}
1 & 0 & 2 & 3 \\
3 & 1 & 5 & -1 \\
5 & 1 & 9 & 2
\end{bmatrix}
\]

State the dimension of each of these spaces and which space $\mathbb{R}^n$ it is a subspace of.

What is the rank of the matrix $A$?
6. Verify that the set

\[ \mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \]

is a basis for \( \mathbb{R}^3 \).

Determine

\[ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{\mathcal{B}} \]

, the coordinate vector for \( \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \) in terms of the basis \( \mathcal{B} \), and then write \( \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \) as a linear combination of the elements of \( \mathcal{B} \).

Write the change-of-coordinates matrix which converts \( \mathcal{B} \)-coordinates to standard coordinates (very easy) and the matrix which converts standard coordinates to \( \mathcal{B} \)-coordinates (takes a tiny bit of work which you can do with your calculator).
7. Hint: your work on this problem should closely resemble work from the previous problem!

Verify that the set \( \{ t + 1, t - 1, t^2 + t, t^3 + t^2 + t + 1 \} \) is a basis for the space \( \mathbb{P}_3 \) of polynomials of degree no greater than two. Your verification should use coordinate vectors for the basis elements with respect to the usual basis for \( \mathbb{P}_3 \). Be sure to both show a calculation and say what it shows about the coordinate vectors.

Find the coordinate vector for the polynomial \( 1 + 2t + 3t^2 + 4t^3 \) with respect to the basis \( \{ t + 1, t - 1, t^2 + t, t^3 + t^2 + t + 1 \} \). Show your calculations, and write \( 1 + 2t + 3t^2 + 4t^3 \) as a linear combination of the elements of the basis.
8. Do one of the following two problems. If you do both your best work will count and if you do well on both you can earn substantial extra credit.

(a) A function from the vector space $\mathbb{P}_2$ of all polynomials of degree no more than two to $\mathbb{R}^2$ is defined by

$$T(p) = \begin{bmatrix} p(1) \\ p(2) \end{bmatrix}$$

Verify that $T$ is a linear transformation.

Describe the kernel of $T$ in purely algebraic terms (without using any terms having to do with vector spaces).
(b) The set of continuous functions from the reals to the reals with the usual definitions of the zero function, addition of functions, and multiplication of functions by real constants is a vector space. We define \( H \) as the set of all continuous functions \( f \) from the reals to the reals with the property that \( f(1) = f(2) \). Show that \( H \) is a subspace of the space of all continuous functions. There are three things to show. Identify each of these three things and prove it.