This exam lasts from 9:40 until 10:35. You may drop one problem, other than problem 7. If you do all problems, your best work will count. Books, notes, and your neighbor must remain firmly closed. You may use any calculator: any operation that you carry out with your calculator (which is restricted to simple calculation, plus finding reduced echelon forms of matrices, matrix products and inverses, and evaluation of determinants) must be set up on your paper.
1. Find the area of the parallelogram with vertices \((0, -2), (6, -1), (-3, 1), (3, 2)\) by setting up and computing a suitable determinant. Show all work.
2. Suppose that \( \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 10 \). Evaluate the following determinants using the properties relating determinants and row (or column) operations. Explain the properties you use.

(a) \( \det \begin{bmatrix} a & b + 2a & c \\ d & e + 2d & f \\ g & h + 2g & i \end{bmatrix} \)

(b) \( \det \begin{bmatrix} d & e & f \\ g & h & i \\ a & b & c \end{bmatrix} \) (be careful!)

(c) \( \det \left( 2 \left( \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \right) \right) \) (again, be careful!)
3. Find bases for the null space, column space, and row space of

\[
\begin{bmatrix}
1 & 2 & 5 & -1 \\
1 & 1 & -3 & 1 \\
3 & 4 & -1 & 1 \\
2 & 2 & 0 & 0 \\
5 & 6 & -1 & 1
\end{bmatrix}
\]

Remember that a basis must be a finite set of vectors.

For each of these spaces, state its dimension and state the space $\mathbb{R}^n$ of which it is a subspace (if it is a subspace of some $\mathbb{R}^n$).
4. Verify that \( f(t), f'(t), f''(t) \) and \( f'''(t) \) make up a basis for \( P_3 \), the space of all polynomials of degree \( \leq 3 \), where \( f(t) = 1 + t + t^2 + t^3 \).

Find the coordinate vector for \( 2 - 9t - 4t^2 - t^3 \) in terms of this basis. (You may arrange the basis elements in whatever order you prefer).

Write down an appropriate expression for \( 2 - 9t - 4t^2 - t^3 \) in terms of the elements of the basis.

Show all calculations and give any explanations that are needed.
5. Verify that
\[ B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \]
is a basis for \( R^3 \).

Give the matrix for conversion from \( B \)-coordinates to standard coordinates. Use this matrix to compute the standard notation for the vector with \( B \)-coordinates \( \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \) (using the order on the basis elements which is given).

Give the matrix for conversion from standard coordinates to the coordinates in terms of this basis. Use this matrix to compute the \( B \)-coordinate vector for \( \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \) (using the given order on the basis elements).
6. There are two neighboring countries on an island. The population of this island remains constant at 20 million. Each year, 10 percent of the population of country A emigrates to country B, while 5 percent of the population of country B emigrates to country A. There are no other significant shifts of population.

Write a stochastic matrix which describes this situation.

If the population of country A this year is 8 million, what will be its population in 5 years?

Find the steady state vector of the stochastic matrix and use it to determine the value toward which the population of country A eventually tends.
7. Determine the eigenvalues and eigenvectors for the matrix

\[
\begin{bmatrix}
-4 & 2 \\
-6 & 4
\end{bmatrix}
\]

Your calculations should use the characteristic polynomial to determine the eigenvalues, then use techniques from section 5.1 to find the eigenvectors for each eigenvalue.

Show calculations that verify directly that each eigenvector you find actually is an eigenvector with the appropriate eigenvalue.
8. Let $P_3$ be the vector space of all polynomials of degree $\leq 3$. One of the following subsets of $P_3$ is a subspace, and one is not. Give an argument verifying that the subset which is a subspace is a subspace (it falls into three parts!) and give a specific example calculation showing how the other fails to be a subspace.

(a) The set of all polynomials of the form $at^3 + bt + c$ – i.e. all the polynomials with no square term.

(b) The set of all polynomials of degree exactly 3.
9. Let $U$ be the vector space of everywhere differentiable functions from the reals to the reals and let $V$ be the vector space of all functions from the reals to the reals.

Verify that the transformation $T$ from $U$ to $V$ which sends each function $f(x)$ in $U$ to the function $f'(x) - f(x)$ in $V$ (i.e., $T(f) = f' - f$) is a linear transformation. This verification is an easy application of properties of the derivative familiar from calculus!

Describe the kernel of this linear transformation (this is an easy calculus problem if you know what “kernel” means).