Notes for Assignment I

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As promised, I'm posting detailed examples similar or identical to things I did in class.

I will put up notes about Monday's quantifier proofs later, probably tomorrow morning.

Example 1: Prove

$$((P \wedge Q) \rightarrow R) \leftrightarrow ((P \rightarrow R) \vee (Q \rightarrow R))$$

This could equally well be written

$$P \wedge Q \rightarrow R \leftrightarrow (P \rightarrow R \vee Q \rightarrow R$$

if we understand the order of operations: $\neg$ is applied first, followed by $\wedge$, then $\vee$, then $\rightarrow$, then $\leftrightarrow$.

Goal:

$$((P \wedge Q) \rightarrow R) \leftrightarrow ((P \rightarrow R) \vee (Q \rightarrow R))$$

[this is a biconditional so the proof has two parts which are proofs of implications in each direction]

Part 1: Assume (1): $((P \wedge Q) \rightarrow R)$

Goal: $((P \rightarrow R) \vee (Q \rightarrow R))$

[we have a strategy for proving disjunctions – assume that one is false and deduce the other]

Assume(2) : $\neg(P \rightarrow R)$

Goal: $Q \rightarrow R$

Assume(3) : Q

Goal 1: R

[we adopt the strategy of proof by contradiction]

Assume (4): $\neg R$

Goal: contradiction

[we have a negative hypothesis $\neg(P \to R)$ above, so we try proving $P \to R$]

Goal 2: $P \to R$

Assume(5): P

Goal: R

we deduce $(6) P \wedge Q$ from (5) and (3).

we deduce $(7) R$ from (1) and (6) by m.p.

This completes the proof of Goal 2 which we can relabel as a conclusion $(8) P \to R$

(2) and (8) give a contradiction, completing the proof of Goal 1, R, and the proof of the goal of Part I by the usual cascade process (it is odd that (4) is never used)

Part 2: Assume (1): $(P \to R) \vee (Q \to R)$

Goal: $(P \wedge Q) \to R$

Assume (2): $P \wedge Q$

Goal: R

[we use proof by cases on assumption (1)]

Case 1: (3) $P \to R$

Goal: R

from (2) we get (4) P.

from (4) and (3) we get R by m.p., which establishes our goal for Case 1.

Case 2: (3) $Q \to R$

Goal: R

from (2) we get (4) Q.

From (3) and (4) we get R by m.p.

Having completed both cases we have completed the proof of Part 2 and so of the whole main goal.

Example 2:

$$\neg(P \vee Q) \leftrightarrow \neg P \wedge \neg Q$$

This is a biconditional so once again we have a proof in two parts.

Part 1: Assume (1): $\neg(P \vee Q)$

Goal: $\neg P \wedge \neg Q$

[to prove a conjunction we prove each part]

Goal 1: $\neg P$

Assume (2): P

Goal: contradiction

[since we have $\neg(P \vee Q)$ as an assumption we try to prove $P \vee Q$]

Goal: $P \vee Q$

Assume (3): $\neg Q$

Goal: P

we already have the Goal as as assumption (2) above (without using (3) at all) so the proof of Goal 1 is complete.

Goal 2: $\neg Q$

Assume (2): Q

Goal: contradiction

[since we have $\neg(P \vee Q)$ as an assumption we try to prove $P \vee Q$]

Goal: $P \vee Q$

Assume (3): $\neg P$

Goal: Q

we already have the Goal as assumption (2) above (without using (3) at all) so the proof of Goal 2 is complete. The proof of Part 1 is complete.

Part 2: Assume (2): $\neg P \wedge \neg Q$

Goal: $\neg(P \vee Q)$

Assume (3): $P \vee Q$

Goal: contradiction

[we use proof by cases on assumption (3)]

Case 1: (4) P

Goal: contradiction

(5) $\neg P$ follows from (2) so we have the desired contradiction between (4) and (5).

Case 2: (4) Q

Goal: contradiction

now (5) $\neg Q$ follows from (2) so we have a contradiction between (2) and (5).

with a contradiction in both cases, we are done.