Notes for Assignment I

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As promised, I’m posting detailed examples similar or identical to things I did in class.

I will put up notes about Monday’s quantifier proofs later, probably tomorrow morning.

Example 1: Prove

\[(P \land Q) \rightarrow R \leftrightarrow ((P \rightarrow R) \lor (Q \rightarrow R))\]

This could equally well be written

\[P \land Q \rightarrow R \leftrightarrow (P \rightarrow R) \lor (Q \rightarrow R)\]

if we understand the order of operations: ¬ is applied first, followed by \&, then \lor, then \rightarrow, then \leftrightarrow.

Goal:

\[((P \land Q) \rightarrow R) \leftrightarrow ((P \rightarrow R) \lor (Q \rightarrow R))\]

[this is a biconditional so the proof has two parts which are proofs of implications in each direction]

Part 1: Assume (1): \((P \land Q) \rightarrow R\)

Goal: \(((P \rightarrow R) \lor (Q \rightarrow R))\)

[we have a strategy for proving disjunctions – assume that one is false and deduce the other]

Assume(2) : \(\neg(P \rightarrow R)\)

Goal: \(Q \rightarrow R\)
Assume(3) : $Q$

**Goal 1:** $R$

[we adopt the strategy of proof by contradiction]

**Assume (4):** $\neg R$

**Goal:** contradiction

[we have a negative hypothesis $\neg(P \rightarrow R)$ above, so we try proving $P \rightarrow R$]

**Goal 2:** $P \rightarrow R$

**Assume(5):** $P$

**Goal:** $R$

we deduce (6) $P \land Q$ from (5) and (3).

we deduce (7) $R$ from (1) and (6) by m.p.

This completes the proof of Goal 2 which we can relabel as a conclusion (8) $P \rightarrow R$

(2) and (8) give a contradiction, completing the proof of Goal 1, $R$, and the proof of the goal of Part I by the usual cascade process (it is odd that (4) is never used)

**Part 2: Assume (1):** $(P \rightarrow R) \lor (Q \rightarrow R)$

**Goal:** $(P \land Q) \rightarrow R$

**Assume (2):** $P \land Q$

**Goal:** $R$

[we use proof by cases on assumption (1)]

**Case 1:** (3) $P \rightarrow R$

**Goal:** $R$

from (2) we get (4) $P$.

from (4) and (3) we get $R$ by m.p., which establishes our goal for Case 1.

**Case 2:** (3) $Q \rightarrow R$

**Goal:** $R$

from (2) we get (4) $Q$.

From (3) and (4) we get $R$ by m.p.

Having completed both cases we have completed the proof of Part 2 and so of the whole main goal.
Example 2:

\[-(P \lor Q) \leftrightarrow \neg P \land \neg Q\]

This is a biconditional so once again we have a proof in two parts.

**Part 1:** Assume (1): \(- (P \lor Q)\)

**Goal:** \(\neg P \land \neg Q\)

[to prove a conjunction we prove each part]

**Goal 1:** \(\neg P\)

Assume (2): \(P\)

**Goal:** contradiction

[since we have \(-(P \lor Q)\) as an assumption we try to prove \(P \lor Q\)]

**Goal:** \(P \lor Q\)

Assume (3): \(\neg Q\)

**Goal:** \(P\)

we already have the Goal as assumption (2) above (without using (3) at all) so the proof of Goal 1 is complete.

**Goal 2:** \(\neg Q\)

Assume (2): \(Q\)

**Goal:** contradiction

[since we have \(-(P \lor Q)\) as an assumption we try to prove \(P \lor Q\)]

**Goal:** \(P \lor Q\)

Assume (3): \(\neg P\)

**Goal:** \(Q\)

we already have the Goal as assumption (2) above (without using (3) at all) so the proof of Goal 2 is complete. The proof of Part 1 is complete.

**Part 2:** Assume (2): \(\neg P \land \neg Q\)

**Goal:** \(\neg (P \lor Q)\)

Assume (3): \(P \lor Q\)

**Goal:** contradiction

[we use proof by cases on assumption (3)]
Case 1: (4) \( P \)
Goal: contradiction
(5) \( \neg P \) follows from (2) so we have the desired contradiction between (4) and (5).

Case 2: (4) \( Q \)
Goal: contradiction
now (5) \( \neg Q \) follows from (2) so we have a contradiction between (2) and (5).

with a contradiction in both cases, we are done.