# Math 314 Homework 3 Solutions

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**original assignment description:** Homework Assignment III : section 1 problems: problem 1 parts i-iii (use as much detail as you can stand); problem 2; problem 5 parts v-ix (you can use the earlier parts, most of which we have done in class); problem 10, problem 12, problem 13, problem 20. If you can do problem 21 or problem 22 I will be impressed. This assignment is due Monday; it is possible that I will grade only parts of it. I suggest reading problem 8 for a more familiar set of axioms for inequalities.

**chapter 1 problem 1:** In all parts of problem 1, the exact details shown in a problem I mark correct will vary from what I show here.

Prove "If $ax = a$ for some number $a \neq 0$ then $x = 1$"

Suppose $a \neq 0$ and $ax = a$.

Then $a^{-1}(ax) = a^{-1}a$ ($a^{-1}$ exists because $a \neq 0$), so

$x =$

(identity of +)

$1x =$

(inverse prop of mult)

$(a^{-1}a)x =$

(assoc prop of mult)

$a^{-1}(ax) =$

(assumption, mult both sides by $a^{-1}$)

$a^{-1}a =$

(inv prop of mult)

1.

$x^2 - y^2 = (x-y)(x+y)$

$(x-y)(x+y) = x(x+y) - y(x+y)$(mult dist over sub, proved in class; I think we proved it on the other side so comm mult is also used)

$= xx + xy - (yx + yy)$

(dist property)

$= xx + xy + -(yx + yy)$

(def sub)

$= xx + xy + -yx + -yy$

(I believe we proved $-(a+b) = -a + -b$ in class)

$= xx + 0 + -yy$

(regrouping and inverse props of +)

$= xx + -yy$(id +)$= x^2 - y^2$

(definition of sub and squares)

If $x^2 - y^2$ then $x = y$ or $x = -y$: Suppose $x^2 = y^2$. Then $x^2 - y^2 = x^2 - x^2 = x^2 + -x^2 = 0$. Thus $(x-y)(x+y) = 0$ (previous part) so $x - y = 0$ or $x + y = 0$ (property of zero proved in class). If $x - y = 0$ then $x + -y = 0$ so $y = 0 + y = x + -y + y = x + 0 = x$. If $x + y = 0$ then $x = -y$ (uniqueness of additive inverses, proved in class).

**problem 2:** The problem is that $x - y = 0$, and the fact that $(x+y)(x-y) = y(x-y)$ does not imply $x + y = y$, because $ac = bc$ only implies $a = b$ if $c \neq 0$.

**problem 5:** **v.** if $a < b$ and $c < 0$ then $ac > bc$

If $b - a$ is positive and $c - 0 = c$ is negative, then $-c$ is positive, $(b-a)(-c)$ is positive, $(b-a)(-c) = ac - bc$ by algebra, so $ac > bc$.

**vi.** if $a > 1$ then $a^2 > a$

If $a > 1$ we have $a > 1 > 0$ so $a > 0$ by transitivity. We can thus multiply both sides of the inequality $a > 1$ by the positive number $a$ to get $a^2 > a$.

**vii.** if $0 < a < 1$ then $a^2 < a$

The assumptions tell us directly that $a$ is positive, so we can multiply both sides of the also given inequality $a < 1$ by the positive $a$ to get $a^2 < a$.

**viii.** if $0 \le a < b$ and $0 \le c < d$ then $ac < bd$

Notice that $a, b, c, d$ are all positive. If we multiply both sides of $a < b$ by the positive $c$ we get $ac < bc$. If we multiply both sides of $c < d$ by the positive $b$ we get $bc < bd$. This give $ac < bc < bd$: we have $ac < bd$ by transitivity.

**ix.** if $0 \le a < b$ then $a^2 < b^2$

By the previous part with $a$ and $c$ both replaced by $a$ and $b$ and $d$ both replaced by $b$.

**problem 10: i.** $|b|$ is $b$ if $b \ge 0$, $-b$ if $b \le 0$.

$|a+b|$ is $a+b$ if $a+b \ge 0$, $-b$ if $a+b \le 0$.

$a+b \ge 0$ iff $b \ge -a$ and $a+b \le 0$ iff $b \le -a$.

So, if $b \ge 0$ and $b \ge -a$, we have $(a+b) - b = a$ as the value of the expression.

If $b \le 0$ and $b \ge -a$, we have $(a+b) + b = a + 2b$ as the value of the expression.

If $b \ge 0$ and $b \le -a$ we have $-(a+b) + b = -a$ as the value of the expression.

If $b \le 0$ and $b \le -a$ we have $-(a+b) - b = -a - 2b$ as the value of the expression.

All the cases are possible.

**ii.** $|x|$ is $x$ if $x \ge 0$ and $-x$ if $x \le 0$.

$|(|x| - 1)|$ is $|x| - 1$ if $|x| \ge 1$ and $1 - |x|$ if $|x| < 1$.

Notice that all four cases are possible.

$x \ge 0$ and $|x| \ge 1$: $x - 1$

$x \ge 0$ and $|x| < 1$: $1 - x$

$x \le 0$ and $|x| \ge 1$: $|x| - 1 = -x - 1$

$x \le 0$ and $|x| \le 1$: $1 - |x| = 1 + x$

**iii.** $|x| - |x^2|$: since $|x^2| = x^2$, this is $x - x^2$ if $x \ge 0$ and $-x - x^2$ if $x \le 0$.

**iv.** $a - |(a - |a|)|$:

$a - |a|$ is 0 if $a \ge 0$ and $-2a$ if $a \le 0$, so $|(a - |a|)|$ is 0 if $a \ge 0$ and $-2a$ if $a \le 0$.

Now, if $a \ge 0$, $a - |(a - |a|)| = a = 0 = a$

while if $a \le 0$, $a - |(a - |a|)| = a - (-2a) = 3a$.

**problem 12:** 1. If $x \geq 0$ and $y \geq 0$, $xy \geq 0$ so $|xy| = xy = |x||y|$
If $x \leq 0$ and $y \leq 0$ then $xy = (-x)(-y) \geq 0$ so $|xy| = xy = (-x)(-y) = |x||y|$.
If $x \geq 0$ and $y \leq 0$ then $x(-y)$ is positive, so $-xy$ is positive, so $xy$ is negative, so $|xy| = -xy = x(-y) = |x||y|$.
The last case $y \geq 0$ and $x \leq 0$ is proved just like the previous one, swapping $x$ and $y$.

2. Goal: $|\frac{1}{x}| = \frac{1}{|x|}$: Since $|x| \cdot |x^{-1}| = |x \cdot x^{-1}| = 1$ by the first part of this question, $|x^{-1}|$ must be the multiplicative inverse of $|x|$, so $|x^{-1}| = (|x|)^{-1}$, which is the same statement.

3. $\frac{|x|}{|y|} = |x| \cdot |y|^{-1} = |x| \cdot |y^{-1}| = |xy^{-1}| = |\frac{x}{y}|$: each step is justified either by the definition of division or by a previous part of this question.

4. $|x - y| = |x + -y| \leq |x| + |-y| = |x| + |y|$

5. Goal: $|x| - |y| \leq |x - y|$: $|x| - |y| = |(x - y) + y| - |y| \leq$(triangle inequality)$(|x - y| + |y|) - |y| = |x - y|$

6. Goal: $|(|x| - |y|)| \leq |x - y|$: $|(|x| - |y|)|$ is equal to either $|x| - |y|$ or $|y| - |x|$, and $|x| - |y| \leq |x - y|$ and $|y| - |x| \leq |y - x| = |x - y|$ both follow from the previous part.

7. $|x+y+z| \leq (1)|x+y|+|z| \leq (2)|x|+|y|+|z|$. Equality holds at (1) iff $x + y$ and $z$ are either both nonpositive or both nonnegative; equality holds in (2) iff $x$ and $y$ are either both nonpositive or both nonnegative. Notice that if both (1) and (2) are equalities and $x$ and $y$ are both nonpositive [nonnegative] then $x + y$ is also nonpositive [nonnegative] and so $z$ is nonpositive [nonnegative]. So the two sides of the original equation are equal iff either all three letters stand for nonpositive numbers or all three letters stand for nonnegative numbers.

**problem 13:** There are two cases, $x \geq y$ and $y \geq x$.

if $y \geq x$ then $\min(x, y) = x$, $\max(x, y) = y$, $|y - x| = y - x$, so $\frac{x+y+|y-x|}{2} = \frac{x+y+y-x}{2} = \frac{2y}{2} = y =\max(x, y)$ and $\frac{x+y-|y-x|}{2} = \frac{x+y-y+x}{2} = \frac{2x}{2} = x =\min(x, y)$.

if $x \geq y$ then $\min(x, y) = y$, $\max(x, y) = x$, $|y - x| = x - y$, so $\frac{x+y+|y-x|}{2} = \frac{x+y+x-y}{2} = \frac{2x}{2} = x =\max(x, y)$ and $\frac{x+y-|y-x|}{2} = \frac{x+y+y-x}{2} =$

$\frac{2y}{2} = y = \min(x, y)$.

I'm not going to try to typeset the expression for $\max(x, y, z)$.

**problem 20:** If $|x - x_0| < \frac{\epsilon}{2}$ and $|y - y_0| < \frac{\epsilon}{2}$, then $|(x+y)-(x_0+y_0)| = |(x - x_0)+(y-y_0)| \leq |x-x_0|+|y-y_0| < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon$, and $|(x-y)-(x_0-y_0)| = |(x-x_0)+(y_0-y)| \leq |x-x_0|+|y_0-y| = |x-x_0|+|y-y_0| < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon$.