Homework 4 Solutions

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ch. 3 problem 21: 1.

$$f \circ (g+h) = f \circ g + f \circ h$$

This is false. Almost any functions will work to give a counterexample.
$f \circ (g+h)(x) = f(g(x)+h(x))$
and
$[f \circ g + f \circ h](x) = f(g(x)) + f(h(x))$
There is no reason to believe these are equal in general.
For example, if $f(x) = x^2$, $g(x) = 2x$ $h(x) = 3x$,
$f \circ (g+h)(1) = f(g(1)+h(1)) = f(2+3) = 25$
while
$[f \circ g + f \circ h](1) = f(g(1)) + f(h(1)) = f(2) + f(3) = 4 + 9 = 13.$

2.

$$(g+h) \circ f = g \circ f + h \circ f$$

This is true. $[(g+h) \circ f](x) = (g+h)(f(x)) = g(f(x)) + h(f(x)) = [g \circ f + h \circ f](x)$

3.

$$\frac{1}{f \circ g} = \frac{1}{f} \circ g$$

$[\frac{1}{f \circ g}](x) = \frac{1}{[f \circ g](x)} = \frac{1}{f(g(x))}$
$[\frac{1}{f} \circ g](x) = [\frac{1}{f}](g(x)) = \frac{1}{f(g(x))}$
This is true.

4.
$$\frac{1}{f \circ g} = f \circ \frac{1}{g}$$

$[\frac{1}{f \circ g}](x) = \frac{1}{[f \circ g](x)} = \frac{1}{f(g(x))}$
from above
$[f \circ \frac{1}{g}](x) = f([\frac{1}{g}](x)) = f(\frac{1}{g(x)})$
and there is no reason to believe this.
With the functions $f(x) = x + 1$ and $g(x) = x + 2$,
$[\frac{1}{f \circ g}](1) = \frac{1}{[f \circ g](1)} = \frac{1}{f(g(1))} = \frac{1}{f(3)} = \frac{1}{4}$
$[f \circ \frac{1}{g}](1) = f([\frac{1}{g}](1)) = f(\frac{1}{g(1)}) = f(\frac{1}{3}) = \frac{4}{3}$
This gives a counterexample.

ch. 4 problem 2: Prove that if $x \in [0, b]$, then $x = tb$ for some t with $0 \leq t \leq 1$.

Assume $x \in [0, b]$, that is $0 \leq x \leq b$. (recall that $b > 0$ is given). Let $t = xb^{-1}$. Notice that $tb = xb^{-1}b = x$, and that $0b^{-1} \leq xb^{-1} \leq bb^{-1}$, that is $o \leq t \leq 1$, because $b^{-1} > 0$, so we can multiply the inequality by b^{-1} and preserve its sense.

Prove that if $x \in [a, b]$, then $x = (1-t)a + tb$ for some t with $0 \leq t \leq 1$.

Assume $x \in [a, b]$ (recall that we assume $a < b$): $a \leq x \leq b$ implies $a - a \leq x - a \leq b - a$, so we have $x - a \in [0, b - a]$, so $x - a = t(b - a)$ for some $t \in [0, 1]$ by the previous paragraph, so $x = (x - a) + a = t(b - a) + a = (1 - t)a + tb$ for the same value of t.

Prove that if (1) $0 \leq t \leq 1$, then $(1 - t)a + tb \in [a, b]$.

Suppose $0 \leq t \leq 1$. $(1-t)a+tb = a+t(b-a)$. (2) $0 \leq t(b-a) \leq b-a$ follows by multiplying both sides of (1) by $b-a > 0$. Adding a to both sides of (2) gives $a \leq a+t(b-a) \leq b$, so $a+t(b-a) = (1-t)a+tb \in [a, b]$.

The points of the open interval are those with t not equal to 0 or 1:

If $t = 0$ or $t = 1$, we have $(1-t)a+tb = a$ or $(1-t)a+tb = b$. We need to show that $(1 - t)a + tb$ cannot be a or b in any other case. Assume $0 < t < 1$. $(1 - t)a + tb = a$ implies $a + t(b - a) = a$, which implies $t(b - a) = 0$, which implies that either $t = 0$ (ruled out) or $b - a = 0$ (ruled out). $(1 - t)a + tb = b$ implies $a + t(b - a) = b$ which implies $t(b-a) = b-a$, which implies $t = 1$ (ruled out). So $(1-t)a+tb \in (a, b)$

ch. 5, 3.iii: The limit is "obviously" 100. $|\frac{100}{x}-100| < \epsilon$ is what we want to make true. $|\frac{100}{x}-100| = |\frac{100-100x}{x}| = 100|\frac{1-x}{x}| = 100\frac{|x-1|}{|x|}$. $100\frac{|x-1|}{|x|} < \epsilon$ would be true if $|x-1| < \frac{\epsilon|x|}{100}$. $\frac{\epsilon|x|}{100}$ does not make sense as δ because it has x in it: we need a lower bound on $|x|$ to make this work. Suppose that $|x-1| < .5$ (we do not choose 1, because our function is undefined at 0, one unit away from $a=1$.) Then we have $.5 < x < 1.5$. Thus we have $|x-1| < \frac{\epsilon(.5)}{100} < \frac{\epsilon|x|}{100}$.

So we set $\delta = \min(.5, \frac{\epsilon(.5)}{100})$.

Assume $0 < |x-1| < \min(.5, \frac{\epsilon(.5)}{100})$. Goal: $|\frac{100}{x}-100| < \epsilon$.

Notice that our assumption also gives us (1) $0 < |x-1| < .5$ and (2) $0 < |x-1| < \frac{\epsilon(.5)}{100}$, and (1) gives us $.5 < x < 1.5$, from which we get (3) $x > .5$.

$|\frac{100}{x}-100| = 100\frac{|x-1|}{|x|} < [by(1)]100\frac{|x-1|}{.5} < [by(2)]100\frac{\frac{\epsilon(.5)}{100}}{.5} = \epsilon$.

ch. 5, 3.iv: Use the identity $x^4 - a^4 = (x-a)(x^3+x^2a+xa^2+a^3)$.

We want to make $|x^4-a^4| < \epsilon$. $|x^4-a^4| = |(x-a)(x^3+x^2a+xa^2+a^3)| = |(x-a)||(x^3+x^2a+xa^2+a^3)|$, so $|x-a| < \frac{\epsilon}{|x^3+x^2a+xa^2+a^3|}$ will do the trick. But $\frac{\epsilon}{|x^3+x^2a+xa^2+a^3|}$ has x in it. What we need is an upper bound on $|x^3+x^2a+xa^2+a^3|$: we cannot enforce this without an upper bound on x. We assume $|x-a| < 1$. It follows that $|x| = |(x-a)+a| \leq |x-a|+|a| < 1+|a|$. Notice that $|a| < 1+|a|$ too; this makes things simpler. Further, $|x^3+x^2a+xa^2+a^3| \leq |x^3|+|x^2a|+|xa^2|+|a^3| < 4(1+|a|)^3$ (replace each x and a in the previous expression with the larger $1+|a|$).

So $|x-a| < \frac{\epsilon}{4(1+|a|)^3} < \frac{\epsilon}{|x^3+x^2a+xa^2+a^3|}$ will work.

We choose $\delta = \min(1, \frac{\epsilon}{4(1+|a|)^3})$.

Suppose $0 < |x-a| < \min(1, \frac{\epsilon}{4(1+|a|)^3})$. Goal: $|x^4-a^4| < \epsilon$.

$|x^4-a^4| = |(x-a)||(x^3+x^2a+xa^2+a^3)| < (\frac{\epsilon}{4(1+|a|)^3})(x^3+x^2a+xa^2+a^3)| < (\frac{\epsilon}{4(1+|a|)^3})(4(1+|a|)^3)$ (because $|x-a| < 1$) and this last expression is obivously equal to ϵ.

ch. 5,3.viii: We want to make $|\sqrt{x}-1| < \epsilon$.

Multiply both sides of this inequality by $\sqrt{x}+1$ and you get $|x-1| < \epsilon(\sqrt{x}+1)$. To get a δ we need a lower bound on x. We assume $|x-1| <$

1. This means $x > 0$, and $(\sqrt{x} + 1) > 1$, so $|x - 1| < \epsilon < \epsilon(\sqrt{x} + 1)$ works.

So $\delta = \min(1, \epsilon)$.

Assume $0 < |x - 1| < \min(1, \epsilon)$. Goal: $|\sqrt{x} - 1| < \epsilon$.

$|\sqrt{x}-1| = \frac{|x-1|}{\sqrt{x}+1} < |x - 1|$, because $|x-1| < 1$, so $x > 0$ and $\sqrt{x}+1 > 1$, and in turn $|x - 1| < \epsilon$, by our choice of δ.