

This document summarizes discussions of basics related to the least upper bound concept and axiom which are not in the book, or maybe in the book but different there from my inclass discussion.

1 The Least Upper Bound: Definitions and Axiom

If A is a set of numbers and b is a number, we say that b is an *upper bound* for A iff for every number $x \in A$ we have $x \leq b$.

A set which has an upper bound is said to be *bounded above*.

An upper bound for A which belongs to A is a *greatest element* of A (or a maximum element). Notice that A can have only one greatest element: suppose b and c are greatest elements of A . It follows that $b \in A$, $c \in A$, and for any $x \in A$, $x \leq b$ and $x \leq c$. From this it follows that $b \leq c$ and $c \leq b$, so $b = c$.

We give completely symmetric definitions of “lower bound” and “least element”.

If A is a set of numbers and b is a number, we say that b is a *lower bound* for A iff for every number $x \in A$ we have $x \geq b$.

A set which has a lower bound is said to be *bounded below*.

A lower bound for A which belongs to A is a *least element* of A (or a minimum element). Notice that A can have only one least element: suppose b and c are least elements of A . It follows that $b \in A$, $c \in A$, and for any $x \in A$, $x \geq b$ and $x \geq c$. From this it follows that $b \geq c$ and $c \geq b$, so $b = c$.

Since we have defined least element of a set, we can define the *least upper bound* of a set A as the least element of the set of upper bounds of A . Since any set has at most one least element, this shows that a set A has just one least upper bound if it has any at all. The least upper bound of A is also called the *supremum* of A . It can be written $\sup(A)$ or $\text{lub}(A)$.

Sets with no upper bound have no least upper bound, of course. The set of all numbers is an example. The empty set has no least upper bound, because *every* number is an upper bound for the empty set.

We give symmetric definitions of greatest lower bound.

Since we have defined greatest element of a set, we can define the *greatest lower bound* of a set A as the greatest element of the set of lower bounds of A . Since any set has at most one greatest element, this shows that a set A has

just one greatest lower bound if it has any at all. The greatest lower bound of A is also called the *infimum* of A . It can be written $\inf(A)$ or $\text{glb}(A)$.

Sets with no lower bound have no greatest lower bound, of course. The set of all numbers is an example. The empty set has no greatest lower bound, because *every* number is a lower bound for the empty set.

The Least Upper Bound axiom says that the exceptions to the existence of least upper bounds exhibited above are the only exceptions (and implies the same thing about greatest lower bounds).

Least Upper Bound Axiom (P13): Any nonempty set which is bounded above has a least upper bound.

This axiom cannot be proved from the other twelve axioms. This is because the system of rational numbers satisfies P1-12, and yet does not satisfy P13. The set of all rational numbers q such that $q^2 < 2$ is bounded above (2 is an upper bound for example) but has no rational least upper bound (it does of course have a least upper bound $\sqrt{2}$ in the system of real numbers). If we assume that the rational numbers are the only numbers, then P1-12 will be true in this system and P13 will not, so P13 cannot be proved from the other axioms.

We prove a theorem which preserves the symmetry of our discussion.

Theorem: Every nonempty set A which is bounded below has a greatest lower bound.

Proof: Let A be a nonempty set of numbers with lower bound b . Define $-A$ as $\{-x \mid x \in A\}$. We claim that $-b$ is an upper bound for $-A$: every element of $-A$ is of the form $-x$ for some $x \in A$, so $x \geq b$, so $-x \leq -b$, which is what we need. Since $-b$ is an upper bound for $-A$, $-A$ is bounded above and so has a least upper bound B . We claim that $-B$ is the greatest lower bound of A : for any $x \in A$, we have $x \geq -B$, because we have $-x \leq B$, because B is an upper bound of $-A$, which shows that $-B$ is an upper bound of A . Now suppose that C is another lower bound of A . $-C$ will be an upper bound of $-A$ (we have already seen how to show this) so $-C \geq B$ (because B is the least upper bound of $-A$) so $C \leq -B$, which shows that $-B$ is the greatest lower bound of A .

The difference between a least upper bound and a maximum element always has to be considered carefully. If a set A has a greatest element (a maximum) then the maximum will be the least upper bound. For example the supremum (least upper bound) of the closed interval $[1, 3]$ is 3, which is also the maximum element. But the open interval $[1, 3)$, which has no largest element, does have a least upper bound, namely 3.

The set $\{1 - \frac{1}{n} \mid n \in \mathbb{N} \wedge n \neq 0\}$ (typical elements of this set are $0, \frac{1}{2}, \frac{2}{3},$ etc.) has no maximum element, but has the least upper bound 1. The limit of a strictly increasing sequence will, as in this case, be the least upper bound of the set of terms of the sequence, and will not be an element of that set.

Notice that any *finite* set of numbers always has a least upper bound, which is also its maximum element, and a greatest lower bound, which is also its minimum element.

Any set of natural numbers or integers which is bounded above has a least upper bound which is also its maximum; any set of integers which is bounded below has a greatest lower bound which is also its minimum.

The statements of the last paragraph are not true if we replace *integer* with *rational number*: we have already seen a set of rational numbers $\{q \in \mathbb{Q} \mid q^2 < 2\}$ which has no maximum element and whose least upper bound is not even a rational number. \mathbb{Q} is a standard notation for the set of rational numbers.

2 Density and the Cut Theorem

The order relations on the rationals and on the reals have a property which distinguishes them from the order relation on the natural numbers and the order relation on the integers:

Observation: For any two rational (real) numbers $x < y$ there is a rational (real) number z such that $x < z < y$.

Proof: Consider $z = \frac{x+y}{2}$.

This is what we mean when we say that the orders on the rationals and the reals are *dense*. There is no “next” rational or real after a given rational or real number.

We certainly expect this to be true because the real numbers “fill” the number line. But the rational numbers do not fill the number line: they are so far from filling the number line that there are more gaps between rational numbers than there are rational numbers...(if you don’t know why this is true, you will by the end of the course; this is quite counterintuitive, I freely admit).

The Least Upper Bound Property is equivalent to a property of the order on the reals much stronger than “density” which is a formalization of the fact that the reals do fill the number line.

A pair of sets of real numbers (L, R) (for Left, Right) is called a “cut” if it satisfies the following conditions:

1. L and R are both nonempty sets.
2. $L \cup R = \mathbb{R}$. The sets L and R together cover the real line.
3. $(\forall x \in L. (\forall y \in R. x < y))$ Each element of L is less than every element of R . Notice that this implies that L and R are disjoint.

A cut is a way of dividing the reals into an upper and lower part, nonoverlapping, covering all the reals.

Theorem: If (L, R) is a cut, then there is a real number x such that either $L = (-\infty, x]$ and $R = (x, \infty)$ or $L = (-\infty, x)$ and $R = [x, \infty)$.

Proof: The set L is nonempty and bounded above by any element of R . So it has a least upper bound $\sup(L)$. Define x as $\sup(L)$. There are two cases: either $x \in L$ or $x \notin L$.

If $x \in L$ we claim that $L = (-\infty, x]$: obviously $L \subseteq (-\infty, x]$ (L is a subset of $(-\infty, x]$) because x is an upper bound for L . We need to show that $(-\infty, x] \subseteq L$. $x \in L$ is given by hypothesis. For any $y < x$, we cannot have $y \in R$, because any element of R has to be greater than $x \in L$ by the definition of a cut. An element y of $(-\infty, x]$ is either less than x or equal to x : we have shown that in both cases $y \in L$. Since we have shown $L \subseteq (-\infty, x]$ and $(-\infty, x] \subseteq L$ we have $L = (-\infty, x]$ in the first case. If $L = (-\infty, x]$ it follows immediately that $R = (x, \infty)$, because $R = \mathbb{R} - L$.

If $x \notin L$ we claim that $L = (-\infty, x)$. Clearly $L \subseteq (-\infty, x)$, because any element of L must be less than $x \in R$. We need to show that

$(-\infty, x] \subseteq L$. This is equivalent to showing that for any $y < x$, we have $y \in L$. Assume $y < x$. y cannot be an upper bound for L , since y is less than the least upper bound x of L . But any element of R is an upper bound for L , so y cannot belong to R , so y must belong to L . Since we have shown $L \subseteq (-\infty, x)$ and $(-\infty, x) \subseteq L$ we have $L = (-\infty, x)$ in the second case. If $L = (-\infty, x)$ it follows immediately that $R = [x, \infty)$, because $R = \mathbb{R} - L$.

The proof is complete.

3 The Least Upper Bound Axiom is Equivalent to the Intermediate Value Theorem

In this section we show that in the presence of the assumptions P1-12 the Least Upper Bound Axiom is equivalent to the more familiar Intermediate Value Theorem. We already know that P1-13 implies the IVT, so all that we need to show is that the Intermediate Value Theorem implies the Least Upper Bound Property in the presence of the other axioms. Further, we prove this by the method of proving the contrapositive: we assume that the Least Upper Bound Property is false, and show that this implies that the Intermediate Value Theorem is false.

We assume that the Least Upper Bound Property is false, that is, that there is a set A of numbers which is nonempty, bounded above, and has no least upper bound.

We define a function f : $f(x) = -1$ for each x which is not a lower bound of A and $f(x) = 1$ for each upper bound of A .

Notice that for any element $a \in A$, for any $x < a$, $f(x) = -1$, and for any upper bound b of A , $f(b) = 1$. Since A is nonempty, there is such an a and such an x , and since A is bounded above, there is such a b . So f takes on the values -1 and 1 (and no others).

We claim that f is continuous. This implies that the Intermediate Value Theorem is false, since we found x and b above such that $x < b$, $f(x) = -1$, $f(b) = 1$, so the IVT would imply $f(c) = 0$ for some $c \in (x, b)$.

We now prove the claim that f is continuous. Fix a real number y . We need to prove $\lim_{x \rightarrow y} f(x) = f(y)$. Fix $\epsilon > 0$. We need to find $\delta > 0$ such that for each x such that $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$.

There are two cases. Either $f(y) = -1$ or $f(y) = 1$.

If $f(y) = 1$, then y is an upper bound for A . Since A has no least upper bound, there is a smaller z which is an upper bound for A . Let $\delta = y - z$: for any x , $|x - y| = |y - x| \leq \delta$ implies $-(y - z) < y - x < y - z$, which implies $x > z$, which implies that x is an upper bound for A , which implies $f(x) = 1$, so $|f(x) - f(y)| = 0 < \epsilon$.

If $f(y) = -1$, then y is not an upper bound for A . Thus there is an element $a \in A$ such that $a > y$. Let $\delta = a - y$. for any x , $|x - y| = |y - x| \leq \delta$ implies $-(a - y) < x - y < a - y$, which implies $x < a$, which implies that x is not an upper bound for A , so $f(x) = -1$, so $|f(x) - f(y)| = 0 < \epsilon$.

In each case we found a suitable δ , so we have proved that f is continuous, and we have seen that this completes the proof.

4 The Nested Interval Theorem (problem 14, section 8)

Definition: A sequence of sets $\{A_i\}_{i \in \mathbb{N}}$ is said to be *nested* iff $A_{i+1} \subseteq A_i$ for each i .

Theorem: A nested sequence of closed intervals has nonempty intersection.

Proof: Let $\{I_i\}_{i \in \mathbb{N}}$ be a nested sequence of closed intervals. There are obviously sequences a and b of numbers such that $I_n = [a_n, b_n]$, and because the I_n 's are nested intervals we have $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$ for each n . More generally, if $i < j$ we have $I_i \subseteq I_j$ so $a_i \leq a_j \leq b_j \leq b_i$. Notice that the a_n 's make up a nondecreasing sequence of numbers and any b_k is an upper bound for all the a_n 's.

The set $A = \{a_n \mid n \in \mathbb{N}\}$ (the set of lower limits of the nested intervals) is nonempty (obviously) and bounded above (by any of the b_k 's), so it has a least upper bound which we will call c . We claim that $c \in I_n$ for any n . To show this we need to show $a_n \leq c \leq b_n$. But this is completely obvious: $c \geq a_n$ because c is an upper bound for the set of all a_n 's, and $c \leq b_n$ because each b_n is an upper bound for the set of a_n 's and c is the least upper bound of this set.