Math 314 Practice Test I

October 6, 2009

Detailed solutions to this practice test will be posted later (Friday afternoon is the plan). I will solve problems on this test on demand on Friday in class.

The whole point of this test is to do the problems entirely on your own, except where I specifically tell you to use resources that I will give you for the test.

Keep an eye on this document: I may add more problems.

1. (propositional logic)

   Write a proof of $(P \wedge Q) \rightarrow R \leftrightarrow ((P \rightarrow R) \vee (Q \rightarrow R))$ in the style presented in class. It's very likely that on the test this sort of question would be an implication rather than a biconditional, or that you would be offered the option of proving one or the other of the implications.

   I will provide a list of proof strategies and rules which you are allowed to use. For the practice test, use my notes.

   Goal: $(P \wedge Q) \rightarrow R \leftrightarrow ((P \rightarrow R) \vee (Q \rightarrow R))$

   This is a biconditional, so the proof breaks into two parts.

   Goal 1: $(P \wedge Q) \rightarrow R \rightarrow ((P \rightarrow R) \vee (Q \rightarrow R))$

   This is an implication.

   Assume (1): $(P \wedge Q) \rightarrow R$

   Goal: $((P \rightarrow R) \vee (Q \rightarrow R))$

   This goal is a disjunction (an or statement). To prove it, assume that one of the alternatives is false and prove that the other follows.

   Assume (2): $\neg(P \rightarrow R)$

Goal: $Q \to R$

This is an implication.

Assume (3): Q

Goal: R

We don't have anything else that applies to this, so we try contradiction.

Assume (4): $\neg R$

Goal: contradiction.

For the sake of a contradiction with $\neg(P \to R)$ assumed above, adopt $P \to R$ as a goal.

Assume(5) : P

Goal: R

We have (6)$P \wedge Q$ by assumptions (3) and (5). We deduce R from (1) and (6) by modus ponens. This completes the proof of Goal 1.

Goal 2: $((\mathrm{P} \to R) \vee (Q \to R)) \to ((P \wedge Q) \to R)$

Assume (1): $((P \to R) \vee (Q \to R))$

Goal: $((P \wedge Q) \to R)$

The new goal is an implication.

Assume (2): $(P \wedge Q)$

Goal: R

We argue by cases using (1):

Case 1: assume (3) $P \to R$

From (2) we have (4) P. From (3) and (4) we get R by modus ponens.

Case 2: assume (3) $Q \to R$

From (2) we have (4) Q. From (3) and (4) we get R by modus ponens.

2. (formal arithmetic)

   Prove the associative law of addition

   $$(x + y) + z = x + (y + z)$$

   from the axioms of formal arithmetic (these axioms will be supplied: for the practice test, use your notes). This is of course an elaborate induction proof. You may not use any theorems, just the axioms.

   I believe that I did this proof in class but have not published it in the notes.

   The actual problem on the test will be one of the theorems we have proved, either in the homework or in class.

3. Prove the inequality $|x| - |y| \leq |x - y|$ from the triangle inequality $|x| + |y| \geq |x + y|$ and algebra. (You may recognize this as one of the parts of problem 12 in the third homework set).

   $|x| = |(x - y) + y| \leq |x - y| + |y|$

   so $|x| - |y| \leq |x - y|$ (by subtracting $|y|$ from both sides of the previous inequality).

4. Prove the inequality $0 \leq x^2$ from basic properties of inequalities (these can be either the axioms or the properties listed in problem 8, section 1).

   Either $x > 0$, $x = 0$ or $x < 0$ (trichotomy).

   This allows a proof by cases.

   Case 1: assume $x > 0$. We can multiply both sides of this inequality by the positive quantity x, getting $x^2 > 0$, so $0 < x^2$, so $0 \leq x^2$.

   Case 2: assume $x = 0$. Then $0 \leq x^2$ because $0 = 0^2$, so $0 \leq 0^2$.

   Case 3: assume $x < 0$. Then we can multiply both sides of this inequality by the positive quantity $-x$, obtaining $-x^2 < 0$, then add x^2 to both sides of the inequality, obtaining $0 < x^2$, so $0 \leq x^2$.

5. Prove that

$$\lim_{x \to 3} x^2 = 9,$$

from the definition of limit, algebra and properties of inequalities (possibly including the triangle inequality) but without using any theorems about limits.

Scratch work: we want to make $|x^2 - 9| < \epsilon$, that is $|x-3||x+3| < \epsilon$, so we want $|x-3| < \frac{\epsilon}{|x+3|}$. $\frac{\epsilon}{|x+3|}$ will not work as δ because it depends on x. We need to restrict values of x so that we have a lower bound on $\frac{\epsilon}{|x+3|}$, which will work if we have an upper bound on $|x+3|$. Let $|x-3| < 1$, so $2 < x < 4$, so $x+3 > 5$ is positive, and $x+3 < 7$. $\frac{\epsilon}{7} < \frac{\epsilon}{|x+3|}$ holds.

So our δ is $\min(1, \frac{\epsilon}{7})$.

Proof: Let $\epsilon > 0$ be an arbitrarily chosen positive real number.

Assume $0 < |x-3| < \min(1, \frac{\epsilon}{7})$. Goal: $|x^2 - 9| < \epsilon$.

Because $|x-3| < 1$, we have $2 < x < 4$, so $x+3 = |x+3|$ (because $x+3 > 5$ is positive) and $|x+3| = x+3 < 7$.

Now $|x^2 - 9| =$ by algebra

$|x-3||x+3| =$ because $x+3$ is positive

$|x-3|(x+3) <$ by bound on $x+3$

$|x-3| \cdot 7 <$ by bound on $|x-3|$

$\frac{\epsilon}{7} \cdot 7 = \epsilon$.

6. Prove that if

$$\lim_{x \to a} f(x) = L$$

and

$$\lim_{x \to a} g(x) = M,$$

then

$$\lim_{x \to a} (f(x) - g(x)) = L - M$$

from the definition of limit, algebra and properties of inequalities (including the triangle inequality) but without using any theorems about limits.