Math 314 Assignment I

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Write proofs in the structured style displayed in class of the following theorems of propositional logic. Do not use "algebraic" rules such as de Morgan's laws to do these exercises.

You may use derived rules such as proof by contrapositive or modus tollens if they have been demonstrated in class.

This assignment is due one week from today (today being Friday the 28th).

1.

$$(P \to Q) \wedge (Q \to R) \to (P \to R)$$

Goal: $(P \to Q) \wedge (Q \to R) \to (P \to R)$

Assume (1): $(P \to Q) \wedge (Q \to R)$

Goal: $P \to R$

Assume (2): P

Goal: R

From (1) we get (3) $P \to Q$
and (4) $Q \to R$
By (2), (3) we get (5) Q by m.p.
By (5), (4) we get R by m.p., which is our goal.

2.

$$((P \vee Q) \to R) \leftrightarrow ((P \to R) \wedge (Q \to R))$$

This is a biconditional, so we prove two implications, one in each direction.

Part 1: Assume (1): $(P \vee Q) \to R)$

Goal: $(P \to R) \wedge (Q \to R)$

This goal breaks into two separate goals, because it is a conjunction.

Goal 1: $(P \to R)$

Assume (2): P

Because we have a hypothesis $(P \vee Q) \to R)$ which is an implication, we try to prove its hypothesis $P \vee Q$ so that we can get R by m.p.

We can conclude (3) $P \vee Q$ because we have P. I *have* mentioned that we are allowed to do this: here is a proof from our basic rules alone:

Goal: $P \vee Q$

Assume: $\neg Q$

Goal: P

we already have the goal because we assumed P above!

From (1), (3) we conclude R by m.p., and this is our goal.

Goal: R

Goal 2: $(Q \to R)$

The proof of this goal is basically the same as that of goal 1.

Assume (2): Q

From Q we can deduce (3) $P \vee Q$ (in basically the same way we deduced it from P above).

From (1) and (3) we deduce R, our goal, by m.p.

Assume (2): Q

Goal: R

Part 2:

Assume (1): $(P \to R) \wedge (Q \to R)$

Goal: $(P \vee Q) \to R)$

Assume (2): $P \vee Q$

Goal: R

From (1) we get (3) $P \to R$ and (4) $Q \to R$.

We use (2) to set up a proof by cases:

Case 1: Assume (5) P

Goal: R

From (5) and (3) we get the goal R by m.p.

Case 2: Assume (5) Q

From (5) and (4) we get the goal R by m.p.

Goal: R

3.

$$\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q)$$

(this is one of de Morgan's laws).

This is another biconditional proof, in two parts.

Part 1: Assume (1): $\neg(P \wedge Q)$

Notice that the only way we have to use a negative assumption or conclusion (other than a double negation) is as part of a contradiction.

Goal: $(\neg P \vee \neg Q)$

Assume (2): $\neg\neg P$

Goal: $\neg Q$

Assume (3): Q

Goal: contradiction

By (2) we have (4) P.

By (3) and (4) we have $P \wedge Q$, which together with (1) gives a contradiction.

Part 2: Assume (1): $(\neg P \vee \neg Q)$

Goal: $\neg(P \wedge Q)$

Assume (2): $P \wedge Q$

Goal: contradiction

From (2) we can get (3) P and (4) Q

Use (1) to make a proof by cases:

Case 1: Assume (5) $\neg P$

Goal: contradiction

The contradiction is immediate from (5) and (3)

Case 2: Assume (5) $\neg Q$

Goal: contradiction

The contradiction is immediate from (5) and (4)

4.

$$\neg(P \to Q) \leftrightarrow (P \wedge \neg Q)$$

Yet another biconditional proof:

Part 1: Assume (1): $\neg(P \to Q)$

Goal: $(P \wedge \neg Q)$

This is a conjunction, so the goal breaks into two subgoals.

Goal 1: P

The only way to prove this seems to be by contradiction.

Assume (2): $\neg P$

Goal: contradiction

We will attempt to prove $P \to Q$ to get a contradiction with (1):

Goal: $P \to Q$

Assume(3): P

Goal: Q

If you have noticed that a contradiction implies anything and (2), (3) are contradictory assumptions, you could just stop here. Using just the basic rules, we don't have to do much more: we prove Q by contradiction:

Assume (4): $\neg Q$

Goal: contradiction

We have a contradiction already from (2) and (3), so we are done.

Goal 2: $\neg Q$

The only way to prove this seems to be by contradiction.

Assume (2): $\neg\neg Q$

Goal: contradiction

We will attempt to prove $P \to Q$ to get a contradiction with (1):

Goal: $P \to Q$

Assume(3): P

Goal: Q

We have Q from (2) above so we are done.

Part 2: Assume (1): $(P \wedge \neg Q)$

Goal: $\neg(P \rightarrow Q)$

Assume (2): $P \rightarrow Q$

Goal: contradiction

From (1) we have (3) P and (4) $\neg Q$

From (1) and (2) we get (5) Q by m.p.

(4) and (5) give the contradiction we need.