

Notes on the Bisection Method

Dr. Holmes

November 6, 2009

Here and in everything that follows, $a < b$ are real numbers (so $[a, b]$ is a closed interval) and f is a continuous function on that closed interval.

1 The Bisection Method in General

We define a sequence of closed intervals $\{I_n\}_{n \in \mathbb{N}}$. Each I_n is of the form $[a_n, b_n]$. $I_1 = [a, b]$, so $a_1 = a$, $b_1 = b$. We assume that for each n , I_{n+1} is either $[a_n, \frac{a_n+b_n}{2}]$ or $[\frac{a_n+b_n}{2}, b_n]$ (which one it is may be different for different values of n): at each step we bisect the interval and choose one of the resulting half-intervals as the next interval.

It is clearly true that $I_{n+1} \subseteq I_n$, so the intersection of all the I_n 's is nonempty, and there is at least one number c which belongs to all the intervals. We claim further that there is exactly one number c which belongs to all the intervals I_n .

Because of the bisection construction, we know that $b_n - a_n = \frac{1}{2^{n-1}}(b - a)$.

Suppose that $c \leq d$ are numbers each of which belongs to all of the intervals I_n . Clearly $0 \leq d - c \leq b_n - a_n$ for each n . Thus $0 \leq d - c \leq \frac{1}{2^{n-1}}(b - a)$ for every n . But this means that $d - c = 0$, so $c = d$.

The bisection method is always to be understood as a way of closing down on a uniquely determined number in $[a, b]$ with desired properties.

2 Using the Bisection Method to Prove the Intermediate Value Theorem

Now suppose that f is continuous on $[a, b]$, $f(a) < 0$ and $f(b) > 0$. Our aim is to find c in (a, b) such that $f(c) = 0$.

All that we need to do is indicate how to choose the I_n 's.

$I_1 = [a, b]$ as usual. We want to choose the I_n 's so that $f(a_n) \leq 0$ and $f(b_n) > 0$ at each stage. (of course if we find $f(a_n) = 0$ we are done!)

We can do this for $n = 1$ by our hypotheses.

Suppose we have chosen I_k so that $f(a_k) \leq 0$ and $f(b_k) > 0$. If $f(\frac{a_k+b_k}{2}) \leq 0$, we choose $[\frac{a_k+b_k}{2}, b_k]$ as I_{k+1} : it satisfies the conditions we want. $f(\frac{a_k+b_k}{2}) > 0$, we choose $[a_k, \frac{a_k+b_k}{2}]$ as I_{k+1} , and it satisfies the conditions we want.

We have shown by mathematical induction that we can find intervals I_k satisfying the desired conditions for every $n \in \mathbb{N}$.

It then follows by the general results about the bisection method that there is a unique real number c which belongs to every I_n .

There are three cases, $c = a$, $c = b$ (these we will show to be impossible) and $c \in (a, b)$. We consider the last of these cases first.

Suppose $f(c) > 0$. Because f is continuous at c , there is a $\delta > 0$ such that if $|x - c| < \delta$, we will have $|f(x) - f(c)| < \frac{|f(c)|}{2}$, from which it follows that $f(x) \geq \frac{|f(c)|}{2} > 0$. But we can find an n such that $\frac{1}{2^{n-1}}(b - a)$, the length of the interval I_n , is less than δ , so the whole interval I_n lies in the interval $(c - \delta, c + \delta)$. We have just seen that all values of f on this interval are positive, but also I_n is included in the interval, and f is nonpositive at a_n by the basic properties of the I_n 's. This is a contradiction.

Suppose $f(c) < 0$. Because f is continuous at c , there is a $\delta > 0$ such that if $|x - c| < \delta$, we will have $|f(x) - f(c)| < \frac{|f(c)|}{2}$, from which it follows that $f(x) \leq -\frac{|f(c)|}{2} < 0$. But we can find an n such that $\frac{1}{2^{n-1}}(b - a)$, the length of the interval I_n , is less than δ , so the whole interval I_n lies in the interval $(c - \delta, c + \delta)$. We have just seen that all values of f on this interval are negative, but also I_n is included in the interval, and f is positive at a_n by the basic properties of the I_n 's. This is a contradiction.

So $f(c) = 0$ in the first case [where $c \in (a, b)$].

Suppose $c = a$ (we need to rule this out, so we assume it and argue to a contradiction). For some $\delta > 0$, for all $x \in [a, a + \delta)$, $|f(x) - f(a)| < \frac{|f(a)|}{2}$, so $f(x) < \frac{f(a)}{2} < 0$. Now choose an I_n with total length $< \delta$ as above, so I_n is included in $[a, a + \delta)$, and get a contradiction in the same way (f is negative on $[a, a + \delta)$ but has a positive value at b_n).

Suppose $c = b$ (we need to rule this out, so we assume it and argue to a contradiction). For some $\delta > 0$, for all $x \in (b - \delta, b]$, $|f(x) - f(b)| < \frac{|f(b)|}{2}$, so $f(x) > \frac{f(b)}{2} > 0$. Now choose an I_n with total length $< \delta$ as above, so I_n is

included in $[a, a + \delta)$, and get a contradiction in the same way. (f is positive on $(b - \delta, b]$ but has a nonpositive value at a_n).

This completes the proof: the only case that can occur is $c \in (a, b)$ and we have shown $f(c) = 0$ in this case.

3 Using the Bisection Method to Prove that a Continuous Function on a Closed Interval is Bounded Above

Suppose that f is continuous on $[a, b]$ and the range of f is unbounded. We deduce a contradiction using the bisection method.

We define the sequence of intervals I_n with the intention that the set of values of f on each I_n have no upper bound.

By hypothesis, this is true for $I_1 = [a, b]$.

Suppose that we have constructed I_k such that the set of values of f on I_k has no upper bound. Then either the set of values of f on $[a_k, \frac{a_k+b_k}{2}]$ is unbounded, in which case we choose $I_{k+1} = [a_k, \frac{a_k+b_k}{2}]$, or the set of values of f on $[a_k, \frac{a_k+b_k}{2}]$ has an upper bound N_1 . It follows in this case that the set of values of f on $[\frac{a_k+b_k}{2}, b_k]$ is unbounded: if it had an upper bound N_2 , then $\max(N_1, N_2)$ would be an upper bound for values of f on I_k , which would contradict our assumptions. So in the second case we can choose $[\frac{a_k+b_k}{2}, b_k]$ as I_{k+1} , and in both cases the set of values of f on I_{k+1} is unbounded. We have shown by mathematical induction that we can select a suitable I_n for every n .

Now we know that there is a unique c which belongs to all the I_n 's. As above, there are three cases, $c = a$, $c = b$, and $c \in (a, b)$. We will show that each one leads to contradiction.

If $c \in (a, b)$, we know that f is continuous at c , so we can choose a δ such that if $|x - c| < \delta$ then $|f(x) - f(c)| < 1$. It follows from this that the set of values of f on $(c - \delta, c + \delta)$ is bounded above by $f(c) + 1$. Now choose n so that the length of I_n is less than δ . The interval I_n will be completely included in $(c - \delta, c + \delta)$ so the values of f on I_n will be bounded as well. This is a contradiction.

If $c = a$, we know that f is continuous at c , so we can choose a δ such that if $a < x < a + \delta$ then $|f(x) - f(a)| < 1$. It follows from this that the set of values of f on $[a, a + \delta)$ is bounded above by $f(c) + 1$. Now choose

n so that the length of I_n is less than δ . The interval I_n will be completely included in $[a, a + \delta)$ so the values of f on I_n will be bounded as well. This is a contradiction.

If $c = b$, we know that f is continuous at c , so we can choose a δ such that if $b - \delta < x < b$ then $|f(x) - f(b)| < 1$. It follows from this that the set of values of f on $(b - \delta, b]$ is bounded above by $f(c) + 1$. Now choose n so that the length of I_n is less than δ . The interval I_n will be completely included in $(b - \delta, b]$ so the values of f on I_n will be bounded as well. This is a contradiction.

4 More Proofs?

More stuff may be added to this document; take a look at it now and then.