

# Manual of Logical Style

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This is the promised (or threatened) document about how to write proofs. It is based on the idea that the logical form of a sentence provides a strong hint as to the appropriate strategy for proving it (if it is a goal) or using it (if it is an assumption).

## 1 A Menagerie of Logical Forms

A mathematical statement is of one of the following forms (usually):

**atomic:** No logical structure: equations or inequalities are examples of atomic statements.

**negation:** “It is not true that  $A$ ”. We often say “not  $A$ ” for brevity but this is not a correct English sentence construction. The logical notation for this is  $\neg A$ . Negation is often absorbed into atomic sentences: we’ll say  $a \neq b$  instead of “ $\neg a = b$ ” and  $a \leq b$  instead of  $\neg a > b$  (for example). Negations of logically complex sentences are very awkward to express in English.

**conjunction:** “ $A$  and  $B$ ”. In formal logical notation, this is written  $A \wedge B$ .

**disjunction:** “ $A$  or  $B$ ”. Remember that in mathematics this usually means “ $A$  or  $B$  or both”; lawyers write “ $A$  and/or  $B$ ” to make the same point clear. In logical notation, this is written  $A \vee B$ .

**implication:** “if  $A$  then  $B$ ”; “ $A$  implies  $B$ ”; “ $B$  if  $A$ ”; “ $A$  is sufficient for  $B$ ”; “ $B$  is necessary for  $A$ ”. There are all kinds of ways to say this because it is important. The logical notation is  $A \rightarrow B$ . It is important to

note that for a mathematician  $A \rightarrow B$  is false just in case  $A$  is true and  $B$  is false, and is true under all other circumstances. In particular,  $A \rightarrow B$  is always true if  $A$  is false. This is not just a convention about symbolism (nothing here is really about symbolism): it applies just as well to the mathematician's uses of the English phrases above.

**biconditional:** “ $A$  if and only if  $B$ ”, “ $A$  iff  $B$ ”. The logical notation is  $A \leftrightarrow B$ .

**universal quantifier:** “for all  $x$ ,  $P(x)$ ”; “for every  $x$ ,  $P(x)$ ”, similarly “any”, “each”. Some of these English words behave differently when nested quantifiers are involved; if we run into trouble with this I'll try to say something about it. The formal notation is  $(\forall x.P(x))$ . Restricted forms like  $(\forall x \in A.P(x))$  or  $(\forall x < n.P(x))$  can be rewritten with an implication:  $(\forall x.x \in A \rightarrow P(x))$  or  $(\forall x.x < n \rightarrow P(x))$ .

**existential quantifier:** “for some  $x$ ,  $P(x)$ ”; “there is  $x$  such that  $P(x)$ ”; “there exists  $x$  such that  $P(x)$ ”. The logical notation is  $(\exists x.P(x))$ . Restricted forms like  $(\exists x \in A.P(x))$  or  $(\exists x < n.P(x))$  can be rewritten as quantified conjunctions:  $(\exists x.x \in A \wedge P(x))$  or  $(\exists x.x < n \wedge P(x))$ .

**disguised quantifiers:** “All men are mortal” is a disguised universally quantified implication: “For all  $x$ , if  $x$  is a man then  $x$  is mortal”. Similarly, “Some unicorns are obnoxious” is a disguised existentially quantified conjunction: “For some  $x$ ,  $x$  is a unicorn and  $x$  is obnoxious”.

**implicit universal quantifiers:** It is very usual in mathematics to conceal universal quantifiers: the statement  $x + y = y + x$  of algebra really has the logical form  $(\forall x.(\forall y.x + y = y + x))$ . [Here I have omitted the intended domain of the quantifiers, which is also something that sometimes happens:  $(\forall x \in \mathbb{R}.(\forall y \in \mathbb{R}.x + y = y + x))$  is better.] This can cause confusion when such a statement is to be negated (whereupon a transformation is usually applied which converts the implicit universal quantifier into an explicit existential one).

**remark on dummy variables:** Just as there is no specific real number  $t$  mentioned in the expression  $\int_0^1 t^2 dt$ , so there is no object  $x$  mentioned in a universally quantified sentence  $(\forall x.P(x))$  or in an existentially quantified sentence  $(\exists x.P(x))$ . This is harder to see somehow for the existential statement, but notice that the  $x$  mentioned in  $(\exists x.P(x))$

may for example not be uniquely determined: in  $(\exists x \in \mathbb{Z}.x^2 = 1)$  it is not correct to say that  $x$  is 1 or to say that it is  $-1$ ; it is not correct to say that it is “1 or  $-1$ ” either, though it is tempting.  $x$  simply does not name any specific number at all.

## 2 Negating Complex Sentences: de Morgan’s Laws

We commented above that it is awkward to negate complex statements in English. In fact, it is usual to automatically transform negations of complex sentences into sentences in which simpler sentences are negated. The trick is to do this correctly.

**negation:** Of course  $\neg\neg A$  is equivalent to  $A$ .

**conjunction:** To say “It is not the case that both  $A$  and  $B$ ” is to say “ $A$  is false *or*  $B$  is false (or both)”. In symbols,  $\neg(A \wedge B)$  is equivalent to  $\neg A \vee \neg B$ .

**disjunction:** To say “It is not the case that either  $A$  or  $B$ ” is to say “ $A$  is false and  $B$  is false”. In symbols,  $\neg(A \vee B)$  is equivalent to  $\neg A \wedge \neg B$ .

**implication:** To say that  $A$  does not imply  $B$  is to say that  $A$  is true and  $B$  is false. This can be confusing when the negated statement is not really strictly an implication: a statement “ $P(x)$  does not (necessarily) imply  $Q(x)$ ” may really be the negation of an implicitly universally quantified implication: in symbols,  $\neg(\forall x.P(x) \rightarrow Q(x))$ , which transforms to  $(\exists x.\neg(P(x) \rightarrow Q(x)))$ , which transforms to  $(\exists x.P(x) \wedge \neg Q(x))$ , that is “for **some**  $x$ ,  $P(x)$  but not  $Q(x)$ ”. [Notice that “but” has the same logical meaning as “and” but here conveys a comment that the second conjunct might be surprising].

**biconditional:** To say that “ $A$  iff  $B$ ” is false is to say that  $A$  is true and  $B$  is false or to say that  $A$  is false and  $B$  is true. It is also equivalent to the exclusive sense of “or”:  $A$  or  $B$  but not both.

**universal quantifier:** To say that it is false that “for all  $x$ ,  $P(x)$ ” is to assert “for some  $x$ ,  $\neg P(x)$ ”. In symbols,  $\neg(\forall x.P(x))$  is equivalent to

$(\exists x.\neg P(x))$ . We have already used this above in the discussion under implication. This is an important principle: to deny a universal statement is to assert the existence of a counterexample.

**existential quantifier:** To say that it is false that “for some  $x$ ,  $P(x)$ ” is to assert “for all  $x$ ,  $\neg P(x)$ ”. In symbols,  $\neg(\exists x.P(x))$  is equivalent to  $(\forall x.\neg P(x))$ .

### 3 General Remarks: Goals and Assumptions

We divide the statements mentioned into a theorem into two classes: *goals* and *assumptions*. A goal is a statement we are trying to prove; an assumption is a statement we are allowed (perhaps just locally to a part of the proof) to assume is true. Our proof strategy falls into two parts: pointers on how to *prove* goals of particular logical forms and pointers on how to *use* assumptions of particular logical forms in proofs.

It is convenient to introduce the name  $\perp$  for a false statement (“the absurd”).

### 4 Proving Statements of Given Logical Forms

In this section we give strategies for proving goals of given forms. One of these (*proof by contradiction*) actually applies to a statement of any form at all.

**negation:** To prove the goal  $\neg A$ , introduce  $A$  as an assumption and take  $\perp$  as the new goal: assume  $A$  and show that an absurdity follows. This is a direct proof of a negative statement: it is not the same thing as proof by contradiction, which is described below. Notice that the assumption  $A$  is local to this part of the proof (if it occurs as a subproof of a larger proof); once  $\perp$  is proved, the assumption  $A$  goes away.

**conjunction:** To prove the goal  $A \wedge B$ , first prove  $A$ , then prove  $B$  (or, if you prefer, first prove  $B$ , then prove  $A$ ).

**implication:** To prove the goal  $A \rightarrow B$ , assume  $A$  then adopt  $B$  as the new goal (note that once  $B$  is proved, the assumption  $A$  goes away; the assumption is local to the proof of the implication).

There is an alternative strategy, which is to prove the equivalent contrapositive statement  $\neg B \rightarrow \neg A$ : assume  $\neg B$  and adopt  $\neg A$  as the new goal. Note that negative assumptions and goals may have the transformations of negative sentences described above applied to them as a matter of course (if you use these transformations, make sure you use them correctly).

To prove the contrapositive is once again not quite the same thing as “proof by contradiction”.

**disjunction:** To prove the goal  $A \vee B$  one could if one were lucky just prove  $A$  or just prove  $B$ , but unfortunately this does not always work. Our official strategy for proving  $A \vee B$  as a goal comes in two symmetrical flavors: either “Assume  $\neg A$  and prove  $B$  as the new goal” or “Assume  $\neg B$  and prove  $A$  as the new goal”.

**biconditional:** To prove the goal  $A \leftrightarrow B$  is to prove the goal  $A \rightarrow B$  then prove the goal  $B \rightarrow A$ . The proof of a biconditional falls naturally into two parts. Further, the contrapositive might be proved instead of one or both of the implications.

A proof outline might look like this: “Assume  $A$ : now prove  $B$ ; assume  $B$ : now prove  $A$ . It is important to notice that the assumption in each part can only be used in that same part. Once  $B$  is proved at the end of the first part, neither the assumption  $A$  nor anything deduced from that assumption in the first part of the proof can be used in the second part of the proof.

Another possible proof outline (using the contrapositive on the second part) is “Assume  $A$ : now prove  $B$ ; assume  $\neg A$ : now prove  $\neg B$ ”.

**proof by contradiction:** To prove a statement  $A$  of arbitrary form, assume  $\neg A$  and aim for the new goal  $\perp$ . Notice that this is the same as proving  $\neg\neg A$  by the strategy for proving a negation.

**universally quantified statement:** To prove  $(\forall x.P(x))$ , let  $a$  be an arbitrary object (about which no additional assumptions can be made) and prove  $P(a)$  as the new goal. To prove  $(\forall x \in A.P(x))$ , let  $a$  be an arbitrary object, assume  $a \in A$ , and prove  $P(a)$  as a new goal. Notice that once this proof is complete, there is no further need to refer to the arbitrary object  $a$  (it is local to this proof).

**existentially quantified statement:** To prove  $(\exists x.P(x))$ , find a specific  $t$  for which you can prove  $P(t)$ . To prove  $(\exists x \in A.P(x))$ , find a specific  $t$  for which you can prove  $t \in A$  and prove  $P(t)$ . Here you actually have to figure out what object  $t$  will work.

$(\exists x.P(x))$  can also be proved by assuming  $(\forall x.\neg P(x))$  and deducing a contradiction. Such a proof of an existential statement is surprising because it does not necessarily tell us what object  $t$  has  $P(t)$  true.

## 5 Using Assumptions of Given Logical Forms

**negation:** If you have shown (or assumed)  $A$  and  $\neg A$ , deduce any proposition  $B$  (in particular, this is the only way that  $\perp$  can be deduced).

**conjunction:** If you have proved or assumed  $A \wedge B$ , you have also shown  $A$  and  $B$  separately.

**disjunction:** If you have proved or assumed  $A \vee B$ , and want to prove goal  $G$ , it is time for the strategy called “proof by cases”: assume  $A$  and prove  $G$ , then assume  $B$  and prove  $G$ , and you have shown that  $G$  follows from  $A \vee B$ .

Also, if you have proved or assumed  $A \vee B$  and also  $\neg A$ , you can conclude  $B$ , and similarly if you have proved or assumed  $A \vee B$  and also  $\neg B$ , you can conclude  $A$ .

**implication:** If you have proved or assumed  $A$  and you have proved or assumed  $A \rightarrow B$ , you have shown  $B$ . This is called *modus ponens*. If you have proved or assumed  $A \rightarrow B$  and you have proved or assumed  $\neg B$ , you can draw the conclusion  $\neg A$ . This is called *modus tollens*.

**biconditional:** You can draw the same conclusions using  $A \leftrightarrow B$  that you can draw from either  $A \rightarrow B$  or  $B \rightarrow A$ .

**universal quantifier:** If you have proved or assumed  $(\forall x.P(x))$  and  $t$  is any specific object, you can draw the conclusion  $P(t)$ . If you have proved or assumed  $(\forall x \in A.P(x))$  and  $t$  is a particular object for which  $t \in A$  has been proved or assumed, then you can conclude  $P(t)$ .

**existential quantifier:** If you have proved or assumed  $(\exists x.P(x))$ , then you can introduce a new object  $c$  with the assumption  $P(c)$  (you are not

allowed to assume anything else about  $c$ ;  $c$  is otherwise arbitrarily chosen). If you have proved or assumed  $(\exists x \in A.P(x))$ , then you can introduce a new object  $c$  with the assumptions  $c \in A$  and  $P(c)$ .