This exam will start at 7:40 am and end at 8:35 am.
No access to books, notes or neighbors is permitted. You are permitted to use a scientific calculator with no symbolic or graphing capabilities (this was announced in class).
1. Classify the following differential equations and initial value problems: for each equation state its order and whether it is linear or nonlinear. In the second part, a “solution” is given (a function or family of functions). Check whether the given family of functions is actually a family of solutions of the equation: I may not be telling the truth! Show all work for your check.

(a) \( x' = \sec(t)x \)

(b) \( x'' = (x')^2 \)
   purported solution: \( x(t) = -\ln(t + c) \)
2. Solve the separable equation.

\[ tx' = x^2 + 1 \]

(a) \( x' = 2x - t; \ x(0) = 1 \)

(b) (extra credit) Solve the linear inhomogeneous equation
\[
v' = \tan(t)v + \cos(t)
\]
4. We analyze the differential equation

\[ x' = (x - 2)(x - 4). \]

Sketch the slope field for this equation. Using the slope field, sketch several solutions, representing the different kinds of qualitative behavior that solutions to this equation can have.

Your sketch should be good enough that I can tell what is happening – it does not have to rival Maple for accuracy!

This equation is autonomous. Identify the equilibrium solutions and draw the phase line. Identify each equilibrium solution as a source, sink, or node, showing all supporting calculations.
5. You need to do just one of the two parts in this question.

In each of the following equations, a change of variables is proposed. Carry out the change of variables, obtaining an equation solving for the new function, and transform the new equation into a form we know how to solve: in each case, identify this form of equation that we know how to solve. It is not necessary to solve the new equation.

(a) \( x' = \frac{\ddot{x}}{t} + \frac{\dot{x}}{\sqrt{t}} \) Use the substitution \( x = ut \).

(b) \( x' = tx - t^2x^2 \) Use the substitution \( x = u^{-1} \).
6. An initial value problem is given:

\[ x' = x^{\frac{1}{3}} + x^{\frac{2}{3}}; x(0) = 0 \]

Tell whether existence of a solution follows from theorems we have studied (this is very different from asking whether you can solve the equation!). If you conclude that there is a solution, tell me whether the existence of just one solution (uniqueness) follows from the theorems we have studied. Explain why, stating clearly what relevant conditions in those theorems hold or do not hold in each case.

Please note that you are not being asked to solve this equation.
7. Numerical methods. You have the choice of doing the theoretical part a or the practical part b (next page).

(a) For what initial value problems

\[ x' = f(x, t); x(t_0) = x_0 \]

does Euler’s method always give precisely correct answers for any \( x(t_1) \)? Your answer should include a precise description of what kind of function \( f(t, x) \) can be, and some kind of supporting argument (your “supporting argument” can be pictorial).
(b) Estimate the value of $x(3)$, where $x$ is the solution to the initial value problem $x' = t + 0.1x; x(2) = 0$, using the Runge-Kutta method with $\Delta x = 0.5$.

If you don’t remember how to do the Runge-Kutta method, you may use the Euler method with $\Delta x = 0.25$ for a modest point penalty (two points out of ten).
8. A tank of water has a capacity of 300 liters. We start with the tank one-third full. We add a 15 percent salt solution at 3 liters/minute while draining the mixture in the tank at 2 liters/minute at the same time. We assume (unrealistically) that the salt is dispersed evenly throughout the mixture as soon as it is added. How much salt is there in the tank at the moment it fills?

Set up the initial value problem to be solved, and explain how you would use the solution to the initial value problem to answer the question. You are not required to actually solve the initial value problem or compute the final answer.