

# Math 333, Summer 2009, Test I

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This exam lasts for our entire class period and ends promptly as the class period ends. You may use a standard scientific calculator without graphing or symbolic computation capabilities. Cell phones must be turned off and out of sight.

Please write your name on both the test paper and your blue book and return both the test paper and the blue book at the end of the exam. No work should be written on the test paper (I will not look for it there!) Please make sure that you remember the number written on the first inside page of your test paper; this will be used to post your grade on the web.

If you have not given me 4 or 5 blue books, please do so at the earliest opportunity.

1. Verify that  $y = \frac{1}{1-e^t}$  is a solution of the differential equation  $y' = y(y-1)$ . Please note that you are not being asked to solve this equation, but to verify that it holds for this particular function.

**Solution:**  $y' = \frac{-1}{(1-e^t)^2}(-e^t) = \frac{e^t}{(1-e^t)^2}$

$$y(y-1) = \left(\frac{1}{1-e^t}\right)\left(\frac{(1-e^t)-1}{1-e^t}\right) = \frac{e^t}{(1-e^t)^2}$$

so,  $y' = y(y-1)$ , which is what we had to check.

2. Sketch the direction field of  $y' = t - y + 1$ . Give slopes at least at all the points with  $t = -2, -1, 0, 1, 2$  and  $y = -2, -1, 0, 1, 2$  (that is 25 different points). Sketch one of the solution curves (there is a natural choice). **Solution:**

I'll sketch this in class.

3. Give the general solution to the separable differential equation  $y' = t(y^2 + 1)$ . Please write  $y$  as a function of  $t$  (i.e., do not leave the solution in implicit form).

**Solution:**

$$y' = t(y^2 + 1)$$

$$\frac{y'}{y^2+1} = t \text{ (no singular solution, as } y^2 + 1 \neq 0)$$

$$\frac{dy}{y^2+1} = t dt$$

$$\arctan(y) = \frac{t^2}{2} + c$$

$$y = \tan\left(\frac{t^2}{2} + c\right)$$

4. Determine the general solution of the separable differential equation  $y' = y^2$  (make sure that you find *all* solutions: there is one exceptional one), then solve the initial value problem  $y' = y^2; y(0) = 1$ . State the interval of existence for this solution.  $y$  should be given explicitly as a function of  $t$ .

$$y' = y^2$$

$$\frac{y'}{y^2} = 1 \text{ (} y = 0 \text{ is a singular solution)}$$

$$\frac{dy}{y^2} = dt$$

$$\frac{y^{-1}}{-1} = t + C$$

$$y^{-1} = -t - C$$

$$y = \frac{1}{-t-C} \text{ (or, if we set } K = -C, \frac{1}{K-t}\text{)}$$

Notice that the singular solution is not included in the solutions  $\frac{1}{K-t}$ .

$y(0) = \frac{1}{K-0} = 1$  gives  $K = 1$ ,  $y(t) = \frac{1}{1-t}$ . The interval of existence is  $(-\infty, 1)$  [the domain of the function also includes the interval  $(1, \infty)$ , but the  $t$  value where we have the initial condition is in the interval  $(-\infty, 1)$ : this is why we choose it].

5. Solve the linear differential equation  $y' = y \tan(t) + t$ .

Change it to the form  $y' - y \tan(t) = t$ .

The integrating factor will be  $e^{\int -\tan(t) dt}$ , that is  $e^{\ln(\cos(t))}$ , that is  $\cos(t)$ .

We multiply both sides of the equation by  $\cos(t)$ :

$$y' \cos(t) - y \sin(t) = t \cos(t)$$

$$(y \cos(t))' = t \cos(t)$$

$$y \cos(t) = \int t \cos(t) dt \text{ set } u = t, v = \sin(t)$$

$$= t \sin(t) - \int \sin(t) dt$$

$$= t \sin(t) + \cos(t) + C$$

since  $y \cos(t) = t \sin(t) + \cos(t) + C$  we have

$$y = t \tan(t) + 1 + C \sec(t)$$

6. A tank contains 100 gallons of salt solution which initially contains 20 pounds of salt. Pure water is added to the solution at 10 gallons/minute, while solution is pumped out at the same rate (keeping the volume constant). Find an expression for the amount of salt in the tank after  $t$  minutes. How long does it take for 90% of the salt to be removed from the tank?

let  $y$  be the amount of salt at time  $t$

$$y(0) = 20$$

$$y' = -\frac{1}{10}y$$

$$y(t) = ce^{-\frac{1}{10}t}$$

$$y(0) = c = 20$$

$$\text{so } y(t) = 20e^{-\frac{1}{10}t}$$

when 90 percent of the salt is out of the tank, ten percent will be left

$$20e^{-\frac{1}{10}t} = 2$$

$$e^{-\frac{1}{10}t} = \frac{1}{10}$$

$$-\frac{1}{10}t = -\ln(10)$$

$$t = 10 \ln(10) \text{ minutes}$$

7. Convert the equation  $y' = -\frac{2xy+2y^2}{x^2+4xy}$  to differential form  $P(x, y)dx + Q(x, y)dy$ . Verify that you get an exact differential equation. Solve it using the technique for exact equations (it happens also to be homogeneous; do not use the technique for homogeneous equations). You may leave the solution in implicit form  $F(x, y) = C$ .

**Solution:**  $(2xy + 2y^2)dx + (x^2 + 4xy)dy = 0$

check for exactness: the partial wrt  $y$  of  $2xy + 2y^2$  is  $2x + 4y$

the partial wrt  $x$  of  $x^2 + 4xy$  is  $2x + 4y$

since these are equal, the equation is exact.

integrate  $2xy + 2y^2$  with respect to  $x$ , treating  $y$  as a constant:  $x^2y + 2xy^2 + f(y)$

integrate  $x^2 + 4xy$  with respect to  $y$ , treating  $x$  as a constant:  $x^2y + 2xy^2 + g(x)$

There are no extra terms with just  $x$  or  $y$ : the solution is  $x^2y + 2xy^2 = C$

8. Solve the homogeneous equation  $(x^2 + y^2)dx - 2x^2dy$  by using the substitution  $y = xv$  then separating the variables. You are solving for  $y$  as a function of  $x$ ; you may leave the solution in implicit form if you wish though it is not especially hard to write an explicit form for  $y$  as it turns out.

I have not done this kind of problem in class: I may talk about it tomorrow and do this as an example.