Math 333 Test Two

Dr. Holmes

February 27, 2003

The test will begin at 7:40 am and end at 8:35 am.
You may use a plain scientific calculator without graphing or symbolic computation capabilities; you may not use any other calculator, and in fact I think there is no need for a calculator at all on this exam.
Books, notes, and neighbors are to remain firmly closed.
Show all work.
The value of each problem is 10 points. Values of parts of some problems are shown.
1. Show (for 6 points) that $y = 0$ and $y = x^5$ are both solutions of the initial value problem

$$y' = 5y^4; y(0) = 0.$$  

Explain why this does not contradict the theorem on uniqueness of solutions to initial value problems. (4 points)
2. Consider the differential equation

\[ y' = (y - 1)(y - 3). \]

Draw the phase line for this equation, identifying and classifying the equilibrium points. Sketch the direction field, including sketches of solution curves of each qualitatively different kind.
3. Convert the differential equation

\[ y'' - y' + y = 0 \]

to a system of two first-order equations in terms of \( y \) and \( v = y' \). (4 points)

Sketch the phase portrait in the \( yv \) plane of this system of equations, providing arrows at least at the points with \( y = -1, 0, 1 \), \( v = -1, 0, 1 \). (6 points)
4. Give the general solutions of the following second-order homogeneous linear equations and initial value problems with constant coefficients. Final answers should not involve complex numbers.

(a) (4 points) \( y'' - 3y' + 2y = 0; \ y(0) = 0; \ y'(0) = 2 \)

(b) (3 points) \( y'' - 4y' + 4y = 0 \)

(c) (3 points) \( y'' - 4y' + 5y = 0 \)
5. Find the general solution to the differential equation $y'' + 3y' + 2y = e^{3t}$. 
6. A circuit contains a coil with inductance 4 faradays and a capacitor with capacitance 1 henry. Suppose the initial voltage across the capacitor is .5 volts and the initial current is 0. Solve for the current at any time $t$. What is the natural frequency of oscillation of the circuit? Hint: you should initially set up and solve an equation for the charge on the capacitor, then differentiate to find the current.
7. Do one of the two following problems. If you do both, your best work will be counted.

(a) Consider a population governed by the logistic equation

\[ P' = 0.1P(50 - P). \]

The variable \( P \) represents the population in thousands of individuals.

i. Describe what will happen if the initial population is 10,000 (i.e, \( P = 10 \))? if it is 25,000? If it is 60,000?

ii. Suppose that 60,000 individuals are removed from this population every time unit. Write the differential equation describing this new situation. Determine and classify its equilibrium points.

iii. Describe what will happen in the new situation if the initial population is 10,000. Describe what will happen if the initial population is 25,000. Describe what happens if the initial population is 60,000.
(b) If you borrow $100,000 at interest of 5.5% per year, compounded continuously, what monthly payment will enable you to pay off your debt in 15 years?
Solve the problem by setting up and solving an appropriate initial value problem. Give your final answer to the nearest cent.