Axioms for Plane Geometry

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Primitive Notion: *point* is an undefined sort of object.

Primitive Notion: A *line* is an undefined sort of collection of points.

Definition: We say a point is on a line when the point is an element of the line.

Definition: The *plane* is the collection of all points.

Primitive Notion: For any line L, we will introduce certain collections of points called *sides* of L.

Primitive Notion: For any point A on a line L, we will introduce certain collections of points called *sides* of A in L.

Axiom 0: There is a line.

Axiom 1: For any two distinct points A and B, there is exactly one line which contains both A and B (which we call line(AB)).

Definition: We say a point A is between points B and C iff A, B, C all lie on the same line L and B and C belong to distinct sides of A in L. When A and B are distinct points, we define segment(AB) as the collection of points between A and B plus the points A and B.

Definition: A set is convex if whenever A and B belong to the set, so does every C between A and B.

Definition: A ray from A is a union of a side of A with $\{A\}$. If it contains B distinct from A, we call it ray(AB).

Axiom 2: For any point A on a line L, there are exactly two sides of A in L, which are nonempty subsets of L. A does not belong to either side. Each point on L other than A belongs to one of the sides of A in L and not the other. If A and B are in the same line L, the side of B in L which does not contain A is a subset of the side of A in L which contains B.

Axiom 3: For any line L, there are exactly two sides of L. The sides and the line are all nonempty sets, and each point belongs to exactly one of these three sets. For any A, B not on L, there is C on L between A and B if and only if A and B are in different sides of L.

Axiom 4 (Continuity Axiom): If a line L is partitioned into two disjoint nonempty convex sets, there will be a point A such that one of the sets is a side of A in L (and so the other will be a ray from A in L).

Primitive Notion: Congruence is a relation between line segments. We write $AB = CD$ to abbreviate "segment(AB) is congruent to segment(CD)". We will define a notion of congruence between angles below.

Axiom 5: Congruence is an equivalence relation on line segments. Note that $AB = BA$ follows from this because segment(AB) and segment(BA) are literally the same set of points.

Axiom 6 (addition of lengths): If B is between A and C and B' is between A' and C' and $AB = A'B'$ and $BC = B'C'$, then $AC = A'C'$.

Axiom 7 (construction of lengths): For any segment BC and any point A and line L, there are exactly two points D and D' on L such that $BC = AD = AD'$, and A is between D and D' (i.e., D and D' are on different sides of A in L).

Definition: We define the angle ABC as the union of ray(BA) and ray(BC) (if the rays are equal, we say it is a zero angle; if the rays are collinear but not equal, we say it is a straight angle). We say that angles ABC and $A'B'C'$ are congruent iff there are points D, E, D', E' in rays BA, BC, $B'A'$, and $B'C'$ such that the segments BD, BE, $B'D'$, $B'E'$ are all congruent and further DE is congruent to $D'E'$.

Axiom 8 (SAS): If $AB = A'B'$, $AC = A'C'$, and angle BAC is congruent to angle $B'A'C'$, then $BC = B'C'$, angle ABC is congruent to angle $A'B'C'$, and angle ACB is congruent to angle $A'C'B'$.

Observation: Use of SAS is needed to see that congruence of angles is an equivalence relation (and doesn't depend in any way on the choice of the points D and E in the definition). It is just what is needed: use of axiom 8 easily shows this.

Definition: The interior of angle ABC (if A, B, C are not collinear) is the intersection of the side of line AB containing C and the side of line CB containing A. If the three points are collinear and B is between A and C, we say that either side of line AB is an interior of the (straight) angle.

Axiom 9 (addition of angles): If angle BAC is congruent to angle $B'A'C'$ and angle CAD is congruent to angle $C'A'D'$, and C is in an interior of angle BAD and C' is in an interior of angle $B'A'D'$, then angle BAD is congruent to angle $B'A'D'$.

Axiom 10: (construction of angles): For any ray AB and nonzero, non-straight angle DEF, there are exactly two rays AC and AC' such that BAC and BAC' are congruent to DEF, and further the rays AC and AC' lie on opposite sides of line AB. A straight angle cannot be congruent to a non-straight angle.

Definition: Two lines are said to be parallel if they do not share a point.

Axiom 11 (parallel postulate): Lines L and M are parallel if there are points A and B on L and M respectively such that for any C on line AB which is not between A and B and which is distinct from both of them, and for D on L and E on M on the same side of line AB, we have angle CBE congruent to angle CAD. Moreover, if lines L and M are parallel, this will always be true for any such A, B, C, D, E.

I believe that I can prove all of Hilbert's axioms from these. In fact, most of these are very closely analogous to Hilbert's axioms. The axiom of angle addition (axiom 9) has no analogue in Hilbert, so I think it can be proved from the others, but I haven't figured out how to do it (I have a general idea how a proof might go, but I don't have all the details).

In particular, I have proved the axioms of sets I and II from axioms 0-3. See if you can do it (some details require you to read carefully!)