Notes for Lecture of Tuesday, Sept. 21

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The reading for this lecture was Bertrand Russell, *Introduction to Mathematical Philosophy*, pp. 1-28. The lecture did not cover all of this material, and it went off on some tangents (clearly marked as such in these notes!) We will continue talking about this reading on Thursday, and I may provide some additional readings as photocopies.

Russell starts by saying what he means by "mathematical philosophy". In brief, he says that when we start with familiar mathematical concepts and develop more complex definitions and theorems, we are engaging in mathematics as usually understood. It is a reverse process, in which we start with familiar mathematical concepts and work toward identifying the proper axioms and primitive notions to use to support these, that Russell calls "mathematical philosophy".

He says that the development of the axioms and common notions of Euclid from the starting point of the practical mathematics familiar to the Greeks (and to the Egyptians and Babylonians) was another example of "mathematical philosophy", while the forward development of the theorems of Euclid from his axioms and primitive notions is an example of "mathematics as usual".

I suggest that the philosopher of mathematics should take an interest in the forward process (the development of new mathematics) as well as in the analytic enterprise of putting everything on firm foundations in which Russell is interested. A modern name for the specific area of philosophy of mathematics which Russell is talking about is "the foundations of mathematics".

Russell makes some claims about the state of mathematics which are important.

1. He claims that all of mathematics had been reduced to the theory of the natural numbers (really, though he did not say this, to the theory

of natural numbers, sets of natural numbers, sets of sets of natural numbers, and so forth). We'll talk about this process of reduction later.

Of course the Pythagoreans had suspected that a reduction to natural number was possible, but the irrationality of the square root of two convinced them to turn to the project of founding everything on geometry instead.

I suggested in response to a student question that this work had been done by Dedekind (defining the reals in terms of natural number arithmetic), by Peano (proposing axioms for the natural numbers), and by Weierstrass (and others) who put the foundations of the calculus on a sound logical footing. We talked about these axioms Tuesday (in fact, we had already started in the previous lecture).

2. He claims that a complete axiomatization of the theory of natural numbers is available (due to Peano). This answers the question of what the axioms for the mathematical theory of natural numbers (and associated sets) should be, but it does not answer the final question...

3. He claims that the question as to what the natural numbers really are (and so what the fundamental objects of mathematics are) had been answered (by Frege). We'll get to this definition on Thursday.

We now introduce the Peano primitive notions and axioms.

Primitive Notions: There are three of these.

One: An object 1. (Russell uses 0 instead).

Successor: A function "successor of": we write $S(x)$ for "the successor of x".

Natural Number: A predicate "is a natural number". We abbreviate this "is a number", since we are not yet talking about any other kind of number.

Axioms: There are five of these.

1. 1 is a number.

2. The successor of a number is a number (for any object n, if n is a number, $S(n)$ is a number).
3. No two numbers have the same successor (for any natural numbers m and n, $S(m) = S(n)$ implies $m = n$).
4. 1 is not the successor of any number (for any natural number n, $S(n) \neq 1$).
5. For any property of natural numbers, if 1 has the property and the successor of any number with the property also has the property, then all natural numbers have the property. (For any property $P(n)$ of natural numbers, if $P(1)$ holds and it is the case that for any natural number n, $P(n)$ implies $P(S(n))$, it follows that $P(m)$ is true for all m).

Axiom 5 is the familiar axiom of mathematical induction.

We proved a theorem in an earlier lecture which I will prove here.

Theorem: For each natural number n, it is the case either that $n = 1$ or that there is a unique m such that $S(m) = n$ (which we call the "predecessor" of n.

Proof: Define a property $P(n)$ as "$n = 1$ or there is a number m such that $S(m) = n$".

$P(1)$ is true, because this means "1=1 or there is a number m such that $S(m) = 1$", and certainly 1=1.

If $P(n)$ is true, so is $P(S(n))$: suppose $P(n)$ is true (though we will actually make no use of this hypothesis!); $P(S(n))$ means "$S(n) = 1$ or there is a natural number m such that $S(m) = S(n)$"; choosing $m = n$ shows that the second alternative is true.

So $P(n)$ has the characteristic that it is true of 1 and true of the successors of numbers of which it is true. Thus, by axiom 5, it is true of all natural numbers n.

Now we know that for any natural number n, either $n = 1$ or there is some m such that $S(m) = n$; we still need to show that this m is unique. Suppose there were numbers m and m' such that $S(m) = n$ and $S(m') = n$; it follows directly from axiom 3 that $m = m'$.

This completes the proof.

It is an important fact about Peano's axioms that they require us to talk about properties of numbers as well as numbers. In the earlier lecture, I phrased axiom 5 in terms of *collections* (or equivalently sets, classes): "any collection of natural numbers which contains 1 and contains $S(n)$ whenever it contains n contains all natural numbers).

It is worth noting here that talk of properties (as abstract objects) can be reduced in mathematics to talk of collections, sets or classes. We certainly think that any property $P(n)$ of natural numbers determines a set $\{n \mid P(n)\}$ of natural numbers, whose elements are exactly the natural numbers which have the property P. "having P" can be translated to "belonging to $\{n \mid P(n)\}$.

The one difference between talk of sets or collections and talk of properties that that we clearly want to identify collections with the same elements, while it is not so clear that we want to identify properties that are true of the same objects. For example, the properties "is a biped without feathers" and "is a rational animal" were thought by Aristotle to be true of the same beings (we are less sure of this now); granting Aristotle his empirical observation, we would conclude that the *sets* of rational animals and of featherless bipeds were the same sets, but we would feel less comfortable with the idea that "being a biped without feathers" and "being a rational animal" were in fact the same property. This discomfort (even if we agreed with Aristotle about the facts) might be taken to arise from the fact that it might be accidental that "being a biped without feathers" and "being a rational animal" hold of the same objects; in mathematics, we are more likely to believe that the facts we discover are necessary truths, so if we find that (for example) sets of natural numbers $\{n \mid P(n)\}$ and $\{n \mid Q(n)\}$ have the same elements(P and Q being distinct sentences expressing a property of a natural number n), we are likely to have found a logical reason why anything with property P must also have property Q and vice versa: this should make it seem reasonable that P and Q are simply the same property described in different ways. This issue will come up again when we talk about founding mathematics on sets.

Russell regards the program of Peano (and the others I mentioned) as founding all of mathematics on the natural numbers in spite of the fact that he clearly also talks about properties of natural numbers or sets of natural numbers because for him sets (he more often says "classes") are part of his logical machinery. We'll talk about this point in more detail later. Notice that Hilbert also talked about sets of points in his axioms for geometry, though "set" was not one of his acknowledged primitive notions.

Now we discuss the definitions of the operations of addition and multiplication (this led us off onto mathematically difficult tangents).

Russell approaches this as follows: for each m, we can define $m+1$ as $S(m)$ (he actually says that he can define $m+0$ as m, since he starts with 0). If we have defined $m+n$, we define $m+S(n)$ as $S(m+n)$.

He says that axiom 5 ensures that we can define $m+n$ for any m and n. The idea is to use the property $P(n)$ defined by "for each natural number m, $m+n$ is defined". This is clearly true for $n=0$, and the form of the definition assures us that if it is true for $n=p$, it will also be true for $n=S(p)$ (suppose we can define $m+p$ for any m; then we can define $m+S(p)$ for any m as $S(m+p)$).

This procedure is suspicious, because mentioning "definability" in our theory in defining our theory itself is known to be dangerous. A brief example: clearly there are only finitely many numbers definable in less than one billion words of English. This means that there must be numbers not definable in less than one billion words, and so there must be a number definable as "the least natural number not definable in less than one billion words of English". But this "definition" uses fewer than one billion words. The conclusion on investigation is that in any formal theory T of natural numbers, it turns out that we cannot define the predicate "n is defined in the language of T by the expression s" in the language of T (at least if we are able to talk about the length of the expression s in the language of T), or we would be able to reproduce this paradox.

Nonetheless, this definition does work. We outline a version that works, but which presents technical problems (which took up too much of our time on Tuesday!)

Define $\texttt{Sum}(m,n,r)$ as follows: $\texttt{Sum}(m,n,r)$ is true if and only if for each ternary relation P which satisfies the following two conditions:

1. for any number x, $P(x,1,S(x))$
2. for any numbers x,y,z, if $P(x,y,z)$, then $P(x,S(y),S(z))$

it is the case that $P(m,n,r)$.

The intention is that $\texttt{Sum}(m,n,r)$ be equivalent to $m+n=r$. We would define $m+n$ as the unique r such that $\texttt{Sum}(m,n,r)$; the technical difficulty is that it is necessary to prove that for each m and n there actually is a unique r such that $\texttt{Sum}(m,n,r)$, and this is not immediately obvious.

We proved just part of this theorem. We expect that if $\texttt{Sum}(m, n, r)$ really means "$m + n = r$" that it should be true that $\texttt{Sum}(m, 1, r)$ should be true if and only if $r = S(m)$ (recalling that $S(m)$ is supposed to mean the same thing as $m + 1$).

Theorem: $\texttt{Sum}(m, 1, r)$ is true if and only if $r = S(m)$.

Proof: There are two things to show.

Goal 1: show that if $r = S(m)$ then $\texttt{Sum}(m, 1, r)$ is true.

Suppose that $r = S(m)$.

If any property P satisfies the two conditions in the definition of $\texttt{Sum}$, then it satisfies $P(x, 1, S(x))$ for any x, so in particular it satisfies $P(m, 1, S(m))$, so it satisfies $P(m, 1, r)$. $\texttt{Sum}(m, 1, r)$ means that any ternary relation P satisfying those properties has $P(m, 1, r)$ true, and we have just shown this.

Goal 2: show that if $\texttt{Sum}(m, 1, r)$ is true then $r = S(m)$.

Suppose that $\texttt{Sum}(m, 1, r)$ is true.

Consider the property $Q(x, y, z)$ defined as "$y \neq 1$ or $z = S(x)$". We claim that this property Q satisfies the conditions on properties given in the definition of $\texttt{Sum}$. Certainly $Q(x, 1, S(x))$ is true: this means "$1 \neq 1$ or $S(x) = S(x)$", and the second alternative is true. Further, we claim that if $Q(x, y, z)$ is true so is $Q(x, S(y), S(z))$, for the trivial reason that $Q(x, S(y), S(z))$ is always true: it means "$S(y) \neq 1$ or $S(z) = S(x)$", and the first alternative is true (by axiom 4). Since Q satisfies the conditions in the definition of $\texttt{Sum}$, $\texttt{Sum}(m, 1, r)$ being true implies $Q(m, 1, r)$, which says "$1 \neq 1$ or $r = S(m)$", which is equivalent to saying $r = S(m)$ (since the first alternative is clearly false). And this is what we wanted to show.

The rest of the argument for the full theorem that $m+n$ is uniquely definable for every m and n (which we did not do in class) is a rather complicated induction, which I may eventually insert here.

Notice that we needed to add talk of relations to talk of properties in our logical machinery in order to handle Russell's definition of addition.

The definition of multiplication (in Russell's style, not our more complex style!) is similar: define $m.1$ as m (Russell defined $m.0$ as 0) and if $m.n$ is defined, define $m.S(n)$ as $m.n + m$.

A student asked whether we could define multiplication directly in terms of addition, using the intuitive idea that $m.n$ is the sum of n copies of m. This doesn't work because we haven't defined the idea of "adding m n times". But we can do this in Russell's style (another digression).

A *function* is a relation R such that for each x, y, and z, if $x\,R\,y$ and $x\,R\,z$, then $y = z$. If R is a function, define $R(x)$ as the unique y such that $x\,R\,y$, if there is such a y.

Now for any function f, define $f^1(x)$ as $f(x)$ (if we started with 0, we would define $f^0(x)$ as x), and define $f^{S(n)}(x)$ as $f(f^n(x))$. This captures the idea of applying the function f n times.

Now we can define addition $m + n$ by $m + n = S^m(n)$ (successor S is a function!) and multiplication almost as easily (just as easily if we had started our numbers with 0 instead of 1): if we had 0 in our system, we would have $m.n = (S^m)^n(0)$, but as it is we must define $m.n$ as $(S^m)^n(1) - 1$ (we have proved that each natural number n other than 1 has a unique predecessor, which we might as well write $n - 1$, and we know that $(S^m)^n(1)$ is not 1, because it is a successor of something. Without using predecessor, we could define $m.1 = m, m.S(n) = (S^m)^n(m)$ (check that this works by doing some examples!)

You should notice that the ability to talk freely about sets, relations, and functions on natural numbers as well as the natural numbers themselves gives us a lot of logical power.

Now we consider the third part of Russell's program (though we don't complete it). We address the question of what the primitive notions of Peano arithmetic actually represent.

The Peano axioms do not give us enough information to determine what its "natural numbers" are. In fact, any progression x_i of objects at all with 1 understood as x_1, successor understood as the map taking each x_n to x_{n+1}, and "natural number" understood as being true only of the x_i's will satisfy the Peano axioms.

An approach which Russell mentions as possible, though he doesn't endorse it, is to interpret every statement of arithmetic as being an assertion not about a particular such progression, but about *all* such progressions: "for any object 1, function S, and property N which satisfy (Peano's axioms), and operation + defined by (our definition of addition), $S(S(1)) + S(S(S(1))) = S(S(S(S(S(S(1))))))$" would express the statement "3+4=7". This statement does in a sense express the fact that 3+4=7, but it doesn't refer to any specific objects as 3,4, or 7 (in fact, the objects playing the roles of 3,4,7 could be any three objects you want in a particular case of this statement: use a progression with three desired objects in third, fourth, and seventh position).

But Russell, while he admits that this position is coherent (it is an example of a "structuralist" approach to mathematics) maintains that there is a definition (due to Frege) which appears to be a reasonable candidate for the "true" definition of the natural numbers. We will talk about this on Thursday. I'll start by reviewing the "structuralist" idea outlined in the previous paragraph, because it is an important alternative view.