Math 187 Assignment VI

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This assignment is due on Wednesday, July 7. I'll probably give out a further assignment with chapter 8 problems during the long weekend or on Tuesday. Problems without specific comments about testworthiness should be considered fair game for the exam.

1. Prove by induction that, for each natural number n, every injection from a set with n elements to a set with n elements is a bijection. The kind of thinking needed for this problem is similar to that needed for the proof of the pigeonhole principle, but it is a much easier problem.

   This is too involved to be a test question.

2. On the set of positive integers less than 20, consider the following two relations: $x\,R\,y$ defined as "$x - y$ is a multiple of 4" and $x\,S\,y$ defined as "$x + y$ is a multiple of 4". One of these is an equivalence relation and one is not.

   For the one which is not an equivalence relation, demonstrate why it is not. For the one which is an equivalence relation, demonstrate that it is, then list the equivalence classes.

3. When I was in school we learned about something called "clock arithmetic" (though it was not introduced in the same way that I will introduce it here!). Maybe some of you did, too. The relation $x\,R\,y$ defined a "$x - y$ is divisible by 12" is an equivalence relation on the integers.

   List the equivalence classes under this relation (there are only finitely many).

   We can define addition, subtraction, and multiplication on these equivalence classes (which are the numbers of clock arithmetic):

if m and n are integers and $[m]$ and $[n]$ are the corresponding equivalence classes, we define

$[m]+[n]=[m+n]$, $[m]-[n]=[m-n]$ and $[m][n]=[mn]$.

Prove that addition and multiplication are well-defined by these definitions: that is, prove that if $m\,R\,m'$ and $n\,R\,n'$ that $m+n\,R\,m'+n'$ and $mn\,R\,m'n'$.

What we are proving here is that the computations $[m]+[n]=[m+n]$ and $[m][n]=[mn]$ are independent of the particular representatives m and n taken from the equivalence classes.

This is very much like an example I did in class! This question is too large to be a test question, but a test question might be similar to part of it.

4. Verify that $x(yz)=(xy)z$ (the associative law of multiplication) holds in our representation of the integers.

5. Verify the "rules of signs": $(+m)(+n)=(+mn)$, $(+m)(-n)=(-mn)$, $(-m)(-n)=(+mn)$, for our representation of the integers. Use the definitions of $+n$ and $-n$ given in class. I did one of the cases in class.

6. For each of the following sets of integers, find an upper bound, a lower bound, the greatest member and the least member, or explain why these objects do not exist:

$$X=\{x\in\mathcal{Z}\mid x^2\geq 50\}$$

$$Y=\{x\in\mathcal{Z}\mid x^2<3x\}$$

7. Exhibit a bijection between the integers and the natural numbers. It would take considerable ingenuity to write a formula for such a bijection: a formal definition by cases isn't too hard, and I will accept a partial list of values for the bijection which shows that you understand the pattern.

   I will give a little extra credit if you *can* write a formula (clever use of $(-1)^n$ *might* do the trick).

   Notice that this means that $\mathcal{N}$ and $\mathcal{Z}$ are infinite sets of the same size.