FROM EULER TO WITTEN
A SHORT SURVEY OF THE VOLUME CONJECTURE IN KNOT THEORY

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Euler 1768

\[-\log(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n}\]

dilogarithm \( L_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} \)

Lobachevsky 1836

hyperbolic geometry

\[H = \{ (x, y, z) \in \mathbb{R}^3 : z > 0 \}\]

\[ds = \sqrt{dx^2 + dy^2 + dz^2} / z\]

geodesic lines: Semicircle or straight line \( \perp \) \( xy \)-plane

geodesic planes: hemisphere or flat plane \( \perp \) \( xy \)-plane

Problem: Calculate volumes of hyperbolic* tetrahedra. difficult!

Coxeter 1935

ideal tetrahedra: vertices in boundary of \( H^3 = xy \)-plane \( \cup \{\infty\} \)

* faces are parts of geodesic planes
similarly class of \( \Delta(\alpha, \beta, \gamma) \)
\[ \alpha + \beta + \gamma = 180 \]

Lobachevsky function
\[
\Lambda(\theta) = \frac{1}{2i} \left( L_2(e^{2i\theta}) - L_2(1) + \theta(\pi - \theta) \right) \\
= - \int_0^\theta \log |\sin u| \, du
\]

Vol \( (\Delta(\alpha, \beta, \gamma)) = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma) \)

Vol \( (\Delta(z)) \)
\[
= \text{im}(L_2(z)) + \arg(z) \log |z|
\]

Isometries of \( \mathbb{H}^3 \):
\[
\text{PSL}(2, \mathbb{C}) = \frac{\text{SL}(2, \mathbb{C})}{\pm I}
\]

Möbius transformations acting on the sphere at infinity
\[ \frac{\mathbb{H}^3}{\text{discrete group of isometrics } \Gamma \equiv \pi_1(M)} \]

\[ M \]

1912

Gieseking manifold: glue faces of an ideal tetrahedron

Riley, Thurston

1975/1978

representations of knot complement

\[ \pi_1(S^3 - K) \rightarrow \text{SL}(2, \mathbb{C}) \]

\[ \text{PSL}(2, \mathbb{C}) \]

K knot

\[ \text{Vol}(S^3 - \mathcal{G}) = 2\text{Vol}(\Delta(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})) = 6 \wedge (\frac{\pi}{3}) = 2.02909832\ldots \]

Mostow, Prasad

1971/1973

RIGIDITY THEOREM: If two hyperbolic manifolds of finite volume, \( n \geq 3 \), have isomorphic fundamental groups then they are isometric.
THURSTON 1: Each knot is a torus knot or a satellite knot or a hyperbolic knot.

THURSTON 2: Each knot complement can be decomposed into finitely many ideal tetrahedra.

If we glue ideal tetrahedra together when will they fit to provide a complete hyperbolic structure?

THURSTON 3: If:
- at each edge several tetrahedra meet. The sum of dihedral angles $= 2\pi$ (Glueing condition)
- pasting of triangles need to get Euclidean boundary torus (Completeness condition)
Jones polynomial of links
\[ J(L; q) \in \mathbb{Z}[q, q^{-1}] \]
\[ J(L; q) = 1 \]
recursively:
\[ q^{-1} J(\uparrow \downarrow; q) - q J(\swarrow \nearrow; q) = (q^{1/2} - q^{-1/2}) J(\swarrow; q) \]

Turaev 1988
\[ N \text{-dim. irreducible representation of } \mathfrak{s}l(2; \mathbb{C}) \]

\[ J_N(L; q) = \text{linear combination of Jones polynomials of } (N-1) \text{-cables of } L \]

R-matrix

Vassiliev 1989
Knot invariants polynomial with respect to crossing change
extend:
\[ I(\uparrow \downarrow) = I(\swarrow \nearrow) - I(\nearrow \searrow) \]
if \( I \equiv 0 \) for \( \geq n \) double points then
\( I \) Vassiliev invariant

Turaev/Viro 1992
Invariants of 3-manifolds by summations over colorings of triangulations using quantized 6j-symbols
quantum dilogarithm
\[ \psi(z) = \prod_{n=1}^{\infty} (1 - z \theta^n), \quad |\theta| < 1 \]
\[ \theta = e^{i \varepsilon} \quad \psi_\varepsilon(z) \sim \frac{1}{\sqrt{1 - z}} \quad e^{\frac{i}{\varepsilon} L_2(z)} \]
for \( \varepsilon \to 0 \)

\[ \theta_N^N = 1 \]
6j-symbols
invariant for triangulated links
in triangulated 3-manifolds
R-matrix

For \( N \geq 1 \):
\[ \langle L \rangle_N \in \mathbb{C} \]

1996
Kashaev conjecture:
For \( K \) a hyperbolic knot,
\[ |\langle K \rangle_N| \] grows exponentially
with growth rate:
\[ \frac{\text{Vol}(S^3 - K)}{2\pi} \]
or
\[ 2\pi \lim_{N \to \infty} \frac{\log |\langle K \rangle_N|}{N} = \text{Vol}(S^3 - K) \]
show: $\langle L \rangle_N = \mathcal{N}(L; e^{\frac{2\pi i}{N}})$

generalize Volume Conjecture to arbitrary knots

Vol $\rightarrow$ simplicial volume

**THEOREM:** Volume Conjecture $\Rightarrow$ Vassiliev Conjecture,
if $K$ is a knot with $\mathcal{N}(K; q) = \mathcal{N}(U; q)$ for $U$ the unknot
then $K$ is the unknot.

**Conjecture:** The asymptotic behavior of the colored Jones polynomial is equal to the perturbative expansion of $SL(2, \mathbb{C})$ gauge theory.

**Volume Conjecture is true for all Volume Zero knots**

"(It is true, as will become clear, if a certain integer-valued coefficient is always nonzero, but we do not know why this would always be so.)"
The Volume Conjecture is a special case of a conjecture relating asymptotically partition functions of Chern-Simons $G$-theory and Chern-Simons $G_C$-theory.

Yokota
Hikami
Garoufalidis/Le
Dylan Thurston
Dijkgraaf/Fuji
$\varphi : \pi_1(S^3 - K) \to SL(2, \mathbb{C})$

$\varphi(\mu) = \begin{bmatrix} m^{\frac{1}{2}} & \ast \\
0 & m^{-\frac{1}{2}} \end{bmatrix}$

$\varphi(\lambda) = \begin{bmatrix} \lambda & \ast \\
0 & \lambda^{-1} \end{bmatrix}$

$(m, \lambda) \in \mathbb{C}^* \times \mathbb{C}^*$

$X_K = \{ (m, \lambda) \in \mathbb{C}^* \times \mathbb{C}^* \mid A_K(m, \lambda) = 0 \}$

$\cong \{ \varphi : \pi_1(S^3 - K) \to SL(2, \mathbb{C}) \}_{\text{hom.}} \left/ \text{conj} \right.$

character variety

\[
\begin{cases}
\hat{A}_K(\hat{m}, \hat{\lambda}; q) \tilde{f}_N(K; q) = 0 \\
\hat{A}_K(\hat{m}, \hat{\lambda}, 1) = A_K(m, \lambda)
\end{cases}
\]

where $\hat{m}, \hat{\lambda}$ (q-Weyl) relations

$\hat{m} f(N) = q^N f(N), \hat{\lambda} f(N) = f(N + 1)$

$q \hat{m} \hat{\lambda} = \hat{\lambda} \hat{m}$

$\left( q = e^{2\pi i / 6} \right)$