FUNCTION SPACES AND SOME RELATIVE COVERING PROPERTIES

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Abstract. *In this paper we defined some relative covering properties of spaces and obtained the relative versions of the basic facts about them.*

1. Introduction. All spaces under consideration are assumed to be Tychonoff topological spaces. By $C_p(X)$ we denote the space of all continuous real-valued functions on a space X in the topology of pointwise convergence. Basic open sets of $C_p(X)$ are of the from $W(f;K;\varepsilon) = \{g \in C_p(X) | \ |g(x) - f(x)| < \varepsilon, \ \forall x \in K, \ K$ is a finite subset of $X\}$.The symbol $\underline{0}$ denotes the constantly zero function.For a subset Y of a space X,the mapping $\pi\colon C_p(X) \to C_p(Y)$ is the restriction mapping i.e. $\pi(f) = f \mid Y$ for every $f \in C_p(X)$. For X a space and for $x \in X$ the symbol Ω_x, denotes the set $\{A \subset X \setminus \{x\} \colon x \in \overline{A}\}$. Our notations and terminology are the same as in [1] and [2].

Recall that a space X is said to have the *Menger property* if for each sequence $(\mathcal{U}_n \colon n \in N)$ of open covers for X there is a sequence $(U_n \colon n \in N)$ such that for each $n \in N$, U_n is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n \in N} U_n$ is a cover of X. A space X has the *Rothberger property* if for each sequence $(\mathcal{U}_n : n \in N)$ of open covers for X there is a sequence $(U_n : n \in N)$ such that for each $n \in N$, U_n is an element of $\mathcal{U}_n$ and $\bigcup_{n \in N} U_n = X$.

An open cover $\mathcal{U}$ of X is said to be an ω-*cover* if $X \notin \mathcal{U}$ and for each finite set $F \subset X$ there is a $U \in \mathcal{U}$ such that $F \subset U$. A space X is said to have ω-*Menger* property if for each sequence $(\mathcal{U}_n : n \in N)$ of ω-covers of X there is a sequence $(U_n : n \in N)$ such that every V_n is a finite subset of U_n and $\bigcup_{n \in N} U_n$ is an ω-cover of X. A space X has the ω-*Rothberger property* if for each sequence $(\mathcal{U}_n : n \in N)$ of ω-covers for X there is a sequence $(U_n : n \in N)$ such that for each $n \in N$, U_n is an element of $\mathcal{U}_n$ and $\bigcup_{n \in N} U_n$ is an ω-cover of X. A space X has *countable fan tightness* if for each $x \in X$ and each sequence $(A_n)_{n \in N}$ of subsets of X such that for each n, $x \in \overline{A_n}$,then there is a sequence $(B_n : n \in N)$ of finite sets such that for each n $B_n \subset A_n$ and $x \in \overline{\bigcup_{n \in N} B_n}$. A space X has *countable strong fan tightness* if for each $x \in X$ and each sequence $(A_n : n \in N)$ of finite sets of X such that $x \in \bigcap_{n \in N} \overline{A}_n$ there exists $x_n \in A_n$ such that $x \in \overline{\{x_n : n \in N\}}$.

Following this terminology we introduce the following definition.

1.1. Definition. Let Y be a subset of a space X. Then:

– Y is said to have *Menger property*(ω-*Menger property*) in X, if for each sequence $(\mathcal{U}_n \colon n \in N)$ of open covers (ω-covers) of X there is a sequence $(U_n \colon n \in N)$ such that for each n, U_n is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n \in N} U_n$ is an open cover (ω-cover)of Y.

– Y is said to have *Rothberger property* (ω-*Rothberger property*) in X if for each sequence $(\mathcal{U}_n : n \in N)$ of open covers (ω-covers)for X there is a sequence $(U_n : n \in N)$ such that for each n,$U_n \in \mathcal{U}_n$ and $\bigcup_{n\in N} U_n$ is an open cover (ω-cover) of Y.

2. Countable fan tightness. If $f\colon X \to Y$ is a continuous mapping,then f has *countable fan tightness* if for each $x \in X$ and each sequence $(A_n : n \in N)$ of elements of Ω_x there is a sequence $(B_n : n \in N)$ of finite sets such that for each n, $B_n \subset A_n$ and $f(x) \in \overline{\bigcup_{n\in N}\ f(B_n)}$.

Theorem 2.1.. *For a space X the following are equivalent:*

(a) *For all n Y^n is Menger in X^n;*

(b) Y is ω-Menger in X;

(c) The mapping π has countable fan tightness.

Proof. $(a) \Rightarrow (b)$: Let $(\mathcal{U}_k : k \in N)$ be a sequence of ω-covers of X and let for each k, $W_k = \{U^n\colon n \in N,\ U \in \mathcal{U}_k\}$. Then every W_k is an open cover of $\Sigma_X = \sum_{n\in N}X^n$ and $({U_n}^k : k \in N)$ is a sequence of open covers of X^n. Since each Y^n has the Menger property in X^n, also $\Sigma_Y = \sum_{n\in N}Y^n$ has the Menger property in Σ_X.Therefore, there is a sequence $(W'_k : k \in N)$ such that for each k,W'_k is a finite subset of W_k and each $y \in \sum_Y$ belongs to $\bigcup W'_l$ for some l.

For each k, let $V_k = \{U \in \mathcal{U}_k$: for some $m, U^m \in W'_k\}$. Then each V_k is a finite subset of $\mathcal{U}_k$ and $\bigcup_{n\in N}V_k$ is an ω-cover of Y in X. Indeed, let $F = \{y_1, y_2, \dots, y_p\}$ be a finite subset of Y. Then $y = (y_1, y_2, \dots, y_p) \in \Sigma_Y$ and so there is a k_0 such that $y \in W_{k_0}$ for some $W_{k_0} \in W'_{k_0}$. But, W_{k_0} is of the form V^p, where $V \in V_{k_0}$, so that $F \subset V$.

$(b) \Rightarrow (a)$: We fix n and let $({\mathcal{U}_n}^k : k \in N)$ be a sequence of open covers of X^n. Let ${V_n}^k$ be the collection of open $V \subset X$ such that V^n is contained in some finite union of elements of ${\mathcal{U}_n}^k$. All ${V_n}^k$ are ω-covers of X. Let F be a finite subset of X and let $U_{F,k}$ be a finite subfamily of U_k such that $F^n \subset \bigcup U_{F,k}$. Since F^n is a compact subset of X^n by a Wallace theorem there is an open set $V \subset X$ such that $F^n \subset V^n \subset \bigcup U_{F,k}$. Therefore, $V \in V_k$ so that V_k is an ω-cover of X. Since Y is ω-Menger in X there is a sequence $(V'_k)_{k\in N}$ such that for each k, V'_k is a finite subset of X and for each finite F of Y there is a $m \in N$ such that one can find a $V \in V'_m$ with $F \subset V$. Let $U_{V,m}$ denote the set of those finitely many elements from $\mathcal{U}_m$ whose union contains V^n; put $U'_m = \bigcup_{V\in V'_m} U_{V,m}$. Then the sequence $(U'_m)_{m\in N}$ witnesses that Y^n has the Menger property in X^n. Indeed, if $y = (y_1, y_2, \dots, y_n)$ is a point in Y^n, then the set $F = \{y_1, y_2, \dots, y_n\}$ is a finite subset of Y and there is an open set $V \in V'_m$ with $F \subset V$. By the constitution of U'_m, there is $m_0 \in N$ such that $V^n \subset U'_{m_0}$, i.e. $y \in U'_{m_0}$.

$(b) \Rightarrow (c)$: Let $(A_n)_{n\in N}$ be a sequence of subsets of $C_p(X)$ the closures of which contain $\underline{0}$. We fix n and for every finite set $F \subset X$ the neighborhood $W = W(\underline{0}; F; \varepsilon)$ intersect A_n so that there exists a function $f_{F,n} \in A_n$ such that $|f_{F,n}(x)| < \varepsilon$ for each $x \in F$. Since $\underline{0}$ and $f_{F,n}$ are continuous functions there are neighborhoods G_x, $x \in F$, such that for $U_F = \bigcup_{x\in F}G_x \supset F$ we have $f_{F,n}(U_F) \subset (-\varepsilon, \varepsilon)$. Let $\mathcal{U}_n = \{U_{F,n}\colon F \in [X]^{<\omega}\}$. For each $n \in N$, $\mathcal{U}_n$ is an ω-cover of X. Since Y is an ω-Menger in X, there exist a sequence $(\mathcal{V}_n : n \in N)$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each finite set $K \subset Y$ there is m such that for some $V \in \mathcal{V}_m$, $K \subset V$. Let $\mathcal{V}_m = \{U_{F_1,m}, \dots, U_{F_{s_m},m}\}$. We take the coresponding functions $f_{F_1,m}, \dots, f_{F_{s_m},m}$ and put $B_m = \{f_{F_1,m}, \dots, f_{F_{s_m},m}\}$. It is clearly that $B_n \subset A_n$, for each $n \in N$.

Let us show that $\pi(\underline{0}) \in \overline{\bigcup_n \pi(B_n)}$. Let $W = W(\pi(\underline{0});\ K;\ \varepsilon)$ be a neighborhood of $\pi(\underline{0})$; let m be a positive integer such that $1/m < \varepsilon$. Since K is a finite subset of Y and Y

is ω-Menger in X there is $p \in N$ such that one can find a $V_p \in \mathcal{V}_p$ such that $K \subset V_p$; let $V_p = U_{F,p}$ for some $i \in \{1, 2, \ldots, s_p\}$. We have $\pi((f_{F_i,p})(K) = f_{F_i,p}(K) \subset f_{F_i,p}(U_{F,p}) \subset (-1/p,\, 1/p) \subset (-1/m, 1/m)$ i.e. $\pi(f_{F_i,p}) \in \pi(B_p) \cap W$.

$(c) \Rightarrow (b)$: Let $(\mathcal{U}_n)_{n \in N}$ be a sequence of ω-covers of X. We fix $n \in N$ and take a finite subset F of X. Let $U_{n,F} = \{U \in \mathcal{U}_n : F \subset U\}$. For each $U \in U_{n,F}$ let $f_{U,F}$ be a continuous function from X into [0,1] such that $f_{U,F}(F) = 0$ and $f_{U,F}(X \setminus U) = 1$. Let $A_n = \{f_{U,F} : F \in [X]^{<\omega},\ U \in U_{n,F}\}$. Then $\underline{0}$ is in the closure of A_n for each $n \in N$. Indeed, let $W(\underline{0}; K; \varepsilon)$ be a basic neighborhood of $\underline{0}$. There is a $U \in \mathcal{U}_n$ which contains K. Then the function $f_{U,K}$ belongs to $A_n \cap W(\underline{0}; K; \varepsilon)$. Since π has countable fan tightness there is a sequence $(B_n)_{n \in N}$ of finite sets such that for each n, $B_n \subset A_n$ and $\pi(\underline{0}) \in \overline{\bigcup_{n \in N} \pi(B_n)}$. Let $\pi(B_n) = \{g_1, \ldots, g_{m_n}\}$ and let $f_{U_i,F_i} \in \pi^{-1}(g_i) \cap B_n$, $i \leq m_n$. We consider the set $\mathcal{V}_n = \{U_1, \ldots, U_{m_n}\} \subset \mathcal{U}_n$. Then the sequence $(V_n)_{n \in N}$ witnesses that Y is ω-Menger in X. Indeed: let K be a finite subset of Y. From $\pi(\underline{0}) \in \overline{\bigcup_{n \in N} \pi(B_n)}$ it follows that $W \cap \pi(B_m) \neq \emptyset$, where $W = W(\pi(\underline{0}); K; 1)$. Fix any $g = \pi(f_{U,F}) \in W \cap \pi(B_m)$. Let us show that $K \subset U$. Suppose not. Then for some $y \in K$ one has $y \notin U$ so that $g(y) = f_{U,F}(y) = 1$, but this contradicts the fact $g \in W$.

3. Countable strong fan tightness. If $f: X \to Y$ is a continuous mapping,then f has *countable strong fan tightness* if for each $x \in X$ and each sequence $(A_n : n \in N)$ of finite sets of X such that $x \in \bigcap_{n \in N} \overline{A_n}$ there exists $x_n \in A_n$ such that $f(x) \in \overline{\{f(x_n) : n \in N\}}$.

We need now the following lemma taken from [5;L.3.2].

Lemma 3.1.. *If $\mathcal{U}$ is an ω-cover of a space X^n, then there is an ω-cover $\mathcal{V}$ of X such that $\{V^n : V \in \mathcal{V}\}$ refines $\mathcal{U}$.*

Theorem 3.2.. *For a space X the following are equivalent:*

(a) *For each $n \in N$,Y^n has the Rothberger property in X^n;*

(b) Y is ω-Rothberger in X;

(c) π has countable strong fan tightness.

Proof. $(a) \Rightarrow (b)$: Let $(\mathcal{U}_n : n \in N)$ be a sequence of ω-covers of X. Let $N = N_1 \cup N_2 \cup \ldots \cup N_n \cup \ldots$ be a partition of N into countably many pairwise disjoint infinite subsets. For every $i \in N$ and every $j \in N_i$ let $V_j = \{U^i : U \in \mathcal{U}_j\}$.Then for every $i \in N$, the sequence $(V_j : j \in N_i)$ is a sequence of ω-covers of X^i.Indeed: let F be a finite subset of X^i, then there is a finite subset G of X such that $F \subset G^i$.Since $\mathcal{U}_j$ is an ω-cover of X,there exist an open set $U \in \mathcal{U}_j$ such that $G \subset U$.Then,$F \subset U^i$,$U \in \mathcal{U}_j$ i.e. $F \subset V_j$. By assumption, for every $i \in N$ one can choose a sequence $(U_j : j \in N_i)$ so that for each j, $U_j \in \mathcal{U}_j$ and $\bigcup_{j \in N_i} {U_j}^i$ is an open cover for Y^i. We shall prove that the sequence $(U_n : n \in N)$ witnesses for $(\mathcal{U}_n : n \in N)$ that Y has the ω-Rothberger property in X.Let $K = \{s_1, s_2, \ldots, s_k\}$ be a finite subset of Y.Then $(s_1, s_2, \ldots, s_k) \in Y^k$ so that there is some $p \in N_k$ such that $(s_1, s_2, \ldots, s_k) \in V_p$. Then $K \subset U_p$.

$(b) \Rightarrow (a)$: We shall prove even more than the assertion claims: if Y has the ω-Rothberger property in X, then for each $n \in N$, Y^n has the ω-Rothberger property in X^n. Let $(\mathcal{U}_k : k \in N)$ be a sequence of ω-covers of X^n. For each $k \in N$ let, according to Lemma 3.1. $\mathcal{V}_k$ be an ω-cover of X such that $\{V^n : V \in \mathcal{V}_k\}$ refines $\mathcal{U}_k$. Using the fact that Y is ω-Rothberger in X we find a sequence $(\mathcal{V}_k : k \in N)$ such that for each k $V_k \in \mathcal{V}_k$ and $\bigcup_{k \in N} V_k \supset Y$. For each k we let U_k denote the element of $\mathcal{U}_k$ for which $V_k^n \subset U_k$. Then $(U_k : k \in N)$ witnesses that Y^n has the ω-Rothberger property in X^n. Let $A = \{a_1, a_2, \ldots, a_s\}$ be a finite subset of Y^n and let for each $i \leq s$, $a_i = (a_{i1}, \ldots, a_{in})$. Then $E = \{a_{ij} : i \leq s, j \leq n\}$ is a finite subset of Y and thus there is m such that $E \subset V_m$.

Then for the set U_m one has $A \subset E^n \subset {V_m}^n \subset U_m$.

$(b) \Rightarrow (c)$: Let $(A_n)_{n\in N}$ be a sequence of subsets of $C_p(X)$ the closures of which contain $\underline{0}$. We set $U_n = \{g^{-1}(-1/n, 1/n) : g \in A_n\}$ for each $n \in N$.For each $n \in N$ and each finite subset F of X a neighborhood $W = W(\underline{0}; F; \varepsilon)$ contains some $g \in A_n$.This means that each U_n is an ω-cover of X.Since Y is an ω-Rothberger in X,for the sequence $\{U_m : m \geq n\}$ of ω-covers, there exist $f_m \in A_m$ such that $\{f_m^{-1}(-1/m, 1/m) : m \geq n\}$ is an ω-cover of Y. Let us show that $\pi(\underline{0}) \in \overline{\{\pi(f_n) : n \in N\}}$. Let $W = W(\pi(\underline{0}); F; \varepsilon)$ be a neighborhood of $\pi(\underline{0})$.There exist $m \geq n$ such that $F \subset f_m^{-1}(-1/m, 1/m)$ and $1/m < \varepsilon$. This means $\pi(\underline{0}) \in \overline{\{\pi(f_m) | m \geq n\}}$.

$(c) \Rightarrow (b)$: Let $(\mathcal{U}_n : n \in N)$ be a sequence of ω-covers of X. We fix $n \in N$ and take a finite subset F of X.We set $A_n = \{f \in C_p(X) : f|_{X\setminus U} = 1 \text{for some} U \in \mathcal{U}_n\}$. Then $\underline{0}$ is in the closure of A_n for each $n \in N$. Indeed, let $W(\underline{0}; K; \varepsilon)$ be a basic neighborhood of $\underline{0}$. There is a $U \in \mathcal{U}_n$ which contains K.By the assumption there exist $f_n \in A_n$ such that $\pi(\underline{0}) \in \overline{\{\pi(f_n) : n \in N\}}$.For each f_n we take $\mathcal{V}_n \in \mathcal{U}_n$ such that $f_n|_{X\setminus\mathcal{V}_n} = 1$. Then the sequence $(\mathcal{V}_n : n \in N)$ witnesses that Y is ω-Rothberger in X. Indeed,let K be a finite subset of Y. We consider the basic open neighborhood $W = W(\pi(\underline{0}); K; 1)$. From $\pi(\underline{0}) \in \overline{\{\pi(f_n) : n \in N\}}$ implies that W contains some $\pi(f_n)$.This means that $K \subset \mathcal{V}_n$. Consequently $(\mathcal{V}_n : n \in N)$ is an ω-cover of Y.

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