FUNCTION SPACES AND SOME RELATIVE COVERING PROPERTIES

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Abstract. In this paper we defined some relative covering properties of spaces and obtained the relative versions of the basic facts about them.

1. Introduction. All spaces under consideration are assumed to be Tychonoff topological spaces. By $C_p(X)$ we denote the space of all continuous real-valued functions on a space $X$ in the topology of pointwise convergence. Basic open sets of $C_p(X)$ are of the form $W(f; K; \varepsilon) = \{g \in C_p(X) \mid |g(x) - f(x)| < \varepsilon, \forall x \in K, \text{K is a finite subset of } X\}$. The symbol $\emptyset$ denotes the constantly zero function. For a subset $A$ of $X$, $A^\omega$ is an $\omega$-cover of $X$ on a space $X$ and for each sequence $(U_n: n \in N)$ of open covers for $X$ there is a sequence $(U_n: n \in N)$ such that for each $n \in N$, $U_n$ is a finite subset of $U_n$ and $\bigcup_{n \in N} U_n$ is a cover of $X$. A space $X$ has the Rothberger property if for each sequence $(U_n: n \in N)$ of open covers for $X$ there is a sequence $(U_n: n \in N)$ such that for each $n \in N$, $U_n$ is an element of $U_n$ and $\bigcup_{n \in N} U_n = X$.

An open cover $\mathcal{U}$ of $X$ is said to be an $\omega$-cover if $X \notin \mathcal{U}$ and for each finite set $F \subset X$ there is a $U \in \mathcal{U}$ such that $F \subset U$. A space $X$ is said to have the $\omega$-Menger property if for each sequence $(U_n: n \in N)$ of $\omega$-covers of $X$ there is a sequence $(U_n: n \in N)$ such that for each $n \in N$, $U_n$ is a finite subset of $U_n$ and $\bigcup_{n \in N} U_n$ is an $\omega$-cover of $X$. A space $X$ has the $\omega$-Rothberger property if for each sequence $(U_n: n \in N)$ of $\omega$-covers of $X$ there is a sequence $(U_n: n \in N)$ such that for each $n \in N$, $U_n$ is an element of $U_n$ and $\bigcup_{n \in N} U_n$ is an $\omega$-cover of $X$. A space $X$ has countable fan tightness if for each $x \in X$ and each sequence $(A_n)_{n \in N}$ of subsets of $X$ such that for each $n$, $x \in \overline{A_n}$, then there is a sequence $(B_n: n \in N)$ of finite sets such that for each $n$, $B_n \subset A_n$ and $x \in \bigcup_{n \in N} B_n$. A space $X$ has countable strong fan tightness if for each $x \in X$ and each sequence $(A_n: n \in N)$ of finite sets of $X$ such that $x \in \bigcap_{n \in N} \overline{A_n}$ there exists $x_n \in A_n$ such that $x \in \{x_n: n \in N\}$.

Following this terminology we introduce the following definition.

1.1. Definition. Let $Y$ be a subset of a space $X$. Then:

- $Y$ is said to have Menger property ($\omega$-Menger property) in $X$, if for each sequence $(U_n: n \in N)$ of open covers (\$\omega\$-covers) of $X$ there is a sequence $(U_n: n \in N)$ such that for each $n$, $U_n$ is a finite subset of $U_n$ and $\bigcup_{n \in N} U_n$ is an open cover (\$\omega\$-cover) of $Y$.

- $Y$ is said to have Rothberger property ($\omega$-Rothberger property) in $X$ if for each
sequence \((\mathcal{U}_n : n \in N)\) of open covers \((\omega\text{-covers})\) for \(X\) there is a sequence \((U_n : n \in N)\) such that for each \(n, U_n \in \mathcal{U}_n\) and \(\bigcup_{n \in N} U_n\) is an open cover \((\omega\text{-cover})\) of \(Y\).

2. Countable fan tightness. If \(f : X \rightarrow Y\) is a continuous mapping, then \(f\) has countable fan tightness if for each \(x \in X\) and each sequence \((A_n : n \in N)\) of elements of \(\Omega_x\) there is a sequence \((B_n : n \in N)\) of finite sets such that for each \(n, B_n \subset A_n\) and \(f(x) \in \bigcup_{n \in N} f(B_n)\).

Theorem 2.1. For a space \(X\) the following are equivalent:

(a) For all \(n Y^n\) is Menger in \(X^n\);
(b) \(Y\) is \(\omega\)-Menger in \(X\);
(c) The mapping \(\pi\) has countable fan tightness.

Proof. (a) \(\Rightarrow\) (b): Let \((\mathcal{U}_k : k \in N)\) be a sequence of \(\omega\)-covers of \(X\) and let for each \(k, W_k = \{U^n : n \in N, U \in \mathcal{U}_k\}\). Then every \(W_k\) is an open cover of \(\Sigma_X = \sum_{n \in N} X^n\) and \((U_n^k : k \in N)\) is a sequence of open covers of \(X^n\). Since each \(Y^n\) has the Menger property in \(X^n\), also \(\Sigma_Y = \sum_{n \in N} Y^n\) has the Menger property in \(\Sigma_X\). Therefore, there is a sequence \((W_k^l : k \in N)\) such that for each \(k, W_k^l\) is a finite subset of \(W_k\) and each \(y \in \Sigma_Y\) belongs to \(\bigcup W_l^l\) for some \(l\).

For each \(k\), let \(V_k = \{U \in \mathcal{U}_k : \text{for some } m, U^m \in W_k^l\}\). Then each \(V_k\) is a finite subset of \(\mathcal{U}_k\) and \(\bigcup_{n \in N} V_k\) is an \(\omega\)-cover of \(Y\) in \(X\). Indeed, let \(F = \{y_1, y_2, \ldots, y_p\}\) be a finite subset of \(F, Y\). Then \(y = (y_1, y_2, \ldots, y_p) \in \Sigma_Y\) and so there is a \(k_0\) such that \(y \in W_{k_0}\) for some \(W_{k_0} \in W_k\). But, \(W_{k_0}\) is of the form \(V^p\), where \(V \in V_{k_0}\), so that \(F \subset V\).

(b) \(\Rightarrow\) (a): We fix \(n\) and let \((U_0^n : k \in N)\) be a sequence of open covers of \(X\). Let \(V^n_k\) be the collection of open \(V \subset X\) such that \(V^n\) is contained in some finite union of \(\omega\)-covers of \(U_0^n\). All \(V^n_k\) are \(\omega\)-covers of \(X\). Let \(F\) be a finite subset of \(X\) and let \(U_{F,k}\) be a finite subfamily of \(U_k\) such that \(F^n \subset \bigcup U_{F,k}\). Since \(F^n\) is a compact subset of \(X^n\) by a Wallace theorem there is an open set \(V \subset X\) such that \(F^n \subset V^n \subset \bigcup U_{F,k}\). Therefore, \(V \in V_k\) so that \(V_k\) is an \(\omega\)-cover of \(X\). Since \(Y\) is \(\omega\)-Menger in \(X\) there is a sequence \((V_k)_{k \in N}\) such that for each \(k, \bigcup V_k\) is a finite subset of \(X\) and for each finite \(F \subset Y\) there is a \(m \in N\) such that one can find a \(V \in V_m\) with \(F \subset V\). Let \(U_{V,m}\) denote the set of those finitely many elements from \(U_m\) whose union contains \(V^n\); put \(U_m^l = \bigcup_{V \in V_m^l} U_{V,m}\).

Then the sequence \((U_m^l)_{m \in N}\) witnesses that \(Y^n\) has the Menger property in \(X^n\). Indeed, if \(y = (y_1, y_2, \ldots, y_n)\) is a point in \(Y^n\), then the set \(F = \{y_1, y_2, \ldots, y_n\}\) is a finite subset of \(Y\) and there is an open set \(V \in V_m^l\) with \(F \subset V\). By the constitution of \(U_m^l\), there is \(m_0 \in N\) such that \(V^n \subset U_{m_0}^l\), i.e. \(y \in U_{m_0}^l\).

(b) \(\Rightarrow\) (c): Let \((A_n)_{n \in N}\) be a sequence of subsets of \(C_p(X)\) the closures of which contain \(\emptyset\). We fix \(n\) and for every finite set \(F \subset X\) the neighborhood \(W = W(\emptyset; F; \varepsilon)\) intersect \(A_n\) so that there exists a function \(f_{F,n} \in A_n\) such that \(|f_{F,n}(x)| < \varepsilon\) for each \(x \in F\). Since \(0\) and \(f_{F,n}\) are continuous functions there are neighborhoods \(G_x, x \in F\) such that for \(U_F = \bigcup_{x \in F} G_x \supset F\) we have \(f_{F,n}(U_F) \subset (0, \varepsilon)\). Let \(U_n = \{U_F, n : F \in [X]^{<\omega}\}\). For each \(n \in N\), \(U_n\) is an \(\omega\)-cover of \(X\). Since \(Y\) is an \(\omega\)-Menger in \(X\), there exist a sequence \((\mathcal{V}_n : n \in N)\) such that for each \(n, \mathcal{V}_n\) is a finite subset of \(U_n\) and for each finite set \(K \subset Y\) there is \(m\) such that for some \(V \in \mathcal{V}_m, K \subset V\). Let \(\mathcal{V}_m = \{f_{F_1,m}, \ldots, f_{F_m,m}\}\).

We take the corresponding functions \(f_{F_1,m}, \ldots, f_{F_m,m}\) and put \(B_m = \{f_{F_1,m}, \ldots, f_{F_m,m}\}\). It is clearly that \(B_n \subset A_n\), for each \(n \in N\).

Let us show that \(\pi(\emptyset) \in \bigcup_{n \in N} \pi(B_n)\). Let \(W = W(\pi(\emptyset); K; \varepsilon)\) be a neighborhood of \(\pi(\emptyset)\); let \(m\) be a positive integer such that \(1/m < \varepsilon\). Since \(K\) is a finite subset of \(Y\) and \(Y\)
is \( \omega \)-Menger in \( X \) there is \( p \in N \) such that one can find a \( V_p \in \mathcal{V}_p \) such that \( K \subseteq V_p \); let \( V_p = U_{F,p} \) for some \( i \in \{1, 2, \ldots, s_p \} \). We have \( \pi((f_{F_i,p})(K)) = f_{F_i,p}(K) \subseteq f_{F_i,p}(U_{F,p}) \subseteq (-1/p, 1/p) \subseteq (-1/m, 1/m) \) i.e., \( \pi(f_{F_i,p}) \in \pi(B_p) \cap W \).

\( \text{(c)} \Rightarrow (b) \): Let \((U_n)_{n \in N}\) be a sequence of \( \omega \)-covers of \( X \). We fix \( n \in N \) and take a finite subset \( F \) of \( X \). Let \( U_{n,F} = \{ U \in U_n : F \subseteq U \} \). For each \( U \in U_{n,F} \) let \( f_{U,F} \) be a continuous function from \( X \) into \( [0, 1] \) such that \( f_{U,F}(F) = 0 \) and \( f_{U,F}(X \setminus U) = 1 \). Let \( A_n = \{ f_{U,F} : F \subseteq [X]^{< \omega}, U \in U_{n,F} \} \). Then \( 0 \) is in the closure of \( A_n \) for each \( n \in N \). Indeed, let \( W(\emptyset ; K; \varepsilon) \) be a basic neighborhood of \( 0 \). There is a \( U \in \mathcal{U}_n \) which contains \( K \). Then the function \( f_{U,K} \in A_n \cap W(\emptyset ; K; \varepsilon) \). Since \( \pi \) has countable fan tightness there is a sequence \((B_n)_{n \in N}\) of finite sets such that for each \( n \), \( B_n \subseteq A_n \) and \( \pi(0) \subseteq \bigcup_{n \in N} \pi(B_n) \).\( \square \)

3. Countable strong fan tightness. If \( f : X \to Y \) is a continuous mapping, then \( f \) has countable strong fan tightness if for each \( x \in X \) and each sequence \((A_n : n \in N)\) of finite sets of \( X \) such that \( x \in \bigcap_{n \in N} \overline{A_n} \) there exists \( x_n \in A_n \) such that \( f(x) \in \{ f(x_n) : n \in N \} \).

We need now the following lemma taken from \textit{[5;L.3.2]}

**Lemma 3.1.** If \( \mathcal{U} \) is an \( \omega \)-cover of a space \( X^n \), then there is an \( \omega \)-cover \( \mathcal{V} \) of \( X \) such that \( \{ V^n : V \in \mathcal{V} \} \) refines \( \mathcal{U} \).

**Theorem 3.2.** For a space \( X \) the following are equivalent:

\( \text{(a)} \) For each \( n \in N \), \( Y^n \) has the Rothberger property in \( X^n \);

\( \text{(b)} \) \( Y \) is \( \omega \)-Rothberger in \( X \);

\( \text{(c)} \) \( \pi \) has countable strong fan tightness.

**Proof.** \( \text{(a)} \Rightarrow (b) \): Let \((U_n : n \in N)\) be a sequence of \( \omega \)-covers of \( X \). Let \( N = N_1 \cup N_2 \cup \ldots \cup N_i \cup \ldots \) be a partition of \( N \) into countably many pairwise disjoint infinite subsets. For every \( i \in N \) and every \( j \in N_i \) let \( V_j = \{ U : U \in \mathcal{U}_j \} \). Then for every \( i \in N \), the sequence \((V_j : j \in N_i)\) is a sequence of \( \omega \)-covers of \( X^i \). Indeed, let \( F \) be a finite subset of \( X^i \), then there is a finite subset \( G \) of \( X \) such that \( F \subseteq G \). Since \( \mathcal{U}_j \) is an \( \omega \)-cover of \( X \), there exist an open set \( U \in \mathcal{U}_j \) such that \( G \subseteq U \). Then \( F \subseteq U \). Indeed, \( F \subseteq U \). By assumption, for every \( i \in N \) one can choose a sequence \((U_j : j \in N_i)\) so that for each \( j \), \( U_j \in \mathcal{U}_j \) and \( \bigcup_{j \in N_i} U_j \) is an open cover for \( Y_i \). We shall prove that the sequence \((U_n : n \in N)\) witnesses that \( Y \) has the \( \omega \)-Rothberger property in \( X \). Let \( K = \{ s_1, s_2, \ldots, s_k \} \) be a finite subset of \( Y \). Then \( (s_1, s_2, \ldots, s_k) \in Y^k \) so that there is some \( p \in N_k \) such that \( (s_1, s_2, \ldots, s_k) \in V_p \). Then \( K \subseteq U_p \).

\( \text{(b)} \Rightarrow (a) \): We shall prove even more than the assertion claims: if \( Y \) has the \( \omega \)-Rothberger property in \( X \), then for each \( n \in N \), \( Y^n \) has the \( \omega \)-Rothberger property in \( X^n \). Let \((U_k : k \in N)\) be a sequence of \( \omega \)-covers of \( X^n \). For each \( k \in N \) let, according to Lemma 3.1, \( V_k \) be an \( \omega \)-cover of \( X \) such that \( \{ V^n : V \in V_k \} \) refines \( U_k \). Using the fact that \( Y \) is \( \omega \)-Rothberger in \( X \) we find a sequence \((V_k : k \in N)\) such that for each \( k \), \( V_k \subseteq V_k \) and \( \bigcup_{k \in N} V_k \supseteq Y \). For each \( k \) we let \( U_k \) denote the element of \( U_k \) for which \( V^n_k \subseteq U_k \). Then \( (U_k : k \in N) \) witnesses that \( Y^n \) has the \( \omega \)-Rothberger property in \( X^n \). Let \( A = \{ a_1, a_2, \ldots, a_n \} \) be a finite subset of \( Y^n \) and let for each \( i \leq s, a_i = (a_{1i}, \ldots, a_{ni}) \). Then \( E = \{ a_{ij} : i \leq s, j \leq n \} \) is a finite subset of \( Y \) and thus there is \( m \) such that \( E \subseteq V_m \).
Then for the set \( U_m \) one has \( A \subset E^n \subset V_m \subset U_m \).

\((b) \Rightarrow (c)\): Let \((A_n)_{n \in \mathbb{N}}\) be a sequence of subsets of \( C_p(X) \) the closures of which contain \( 0 \). We set \( U_n = \{g^{-1}(-1/n,1/n) : g \in A_n\} \) for each \( n \in \mathbb{N} \). For each \( n \in \mathbb{N} \) and each finite subset \( F \) of \( X \) a neighborhood \( W = W(0;F;\varepsilon) \) contains some \( g \in A_n \). This means that each \( U_n \) is an \( \omega \)-cover of \( X \). Since \( Y \) is an \( \omega \)-Rothberger in \( X \), for the sequence \( \{U_m : m \geq n\} \) of \( \omega \)-covers, there exist \( f_m \in A_m \) such that \( \{f_m^{-1}(-1/m,1/m) : m \geq n\} \) is an \( \omega \)-cover of \( Y \). Let us show that \( \pi(0) \in \{\pi(f_n) : n \in \mathbb{N}\} \). Let \( W = W(\pi(0);F;\varepsilon) \) be a neighborhood of \( \pi(0) \). There exist \( m \geq n \) such that \( F \subset f_m^{-1}(-1/m,1/m) \) and \( 1/m < \varepsilon \). This means \( \pi(0) \in \{\pi(f_m) : m \geq n\} \).

\((c) \Rightarrow (b)\): Let \((U_n : n \in \mathbb{N})\) be a sequence of \( \omega \)-covers of \( X \). We fix \( n \in \mathbb{N} \) and take a finite subset \( F \) of \( X \). We set \( A_n = \{f \in C_p(X) : f|_{X \setminus U} = 1 \text{ for some } U \in U_n\} \). Then \( 0 \) is in the closure of \( A_n \) for each \( n \in \mathbb{N} \). Indeed, let \( W(0;K;\varepsilon) \) be a basic neighborhood of \( 0 \). There is a \( U \in U_n \) which contains \( K \). By the assumption there exist \( f_0 \in A_n \) such that \( \pi(0) \in \{\pi(f_n) : n \in \mathbb{N}\} \). For each \( f_n \) we take \( V_n \in U_n \) such that \( f_n|_{X \setminus V_n} = 1 \). Then the sequence \( \langle V_n : n \in \mathbb{N}\rangle \) witnesses that \( Y \) is \( \omega \)-Rothberger in \( X \). Indeed, let \( K \) be a finite subset of \( Y \). We consider the basic open neighborhood \( W = W(\pi(0);K;\varepsilon) \). From \( \pi(0) \in \{\pi(f_n) : n \in \mathbb{N}\} \) implies that \( W \) contains some \( \pi(f_n) \). This means that \( K \subset V_n \). Consequently \( \langle V_n : n \in \mathbb{N}\rangle \) is an \( \omega \)-cover of \( Y \).

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