ON RELATIVE $\gamma$-SETS

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Abstract. In this note we show a relative version of the Gerlits-Nagy theorem concerning a characterization of $\gamma$-sets in terms of function spaces. Our result involves a property of the corresponding mapping between function spaces.

1. Introduction

In this note $X$ will denote a Tychonoff space and $Y$ will be a subspace of $X$. The notation and terminology we follow are standard [E]. An open cover $U$ of a space $X$ is an $\omega$-cover [GN] if $X$ does not belong to $U$ and every finite subset of $X$ is contained in an element of $U$. An open cover $U$ of $X$ is called a $\gamma$-cover [GN] if it is infinite and for each $x \in X$ the set $\{U \in U : x \notin U\}$ is finite. Let us observe that every $\gamma$-cover is an $\omega$-cover; moreover, each finite subset of a space belongs to all but finitely many elements of a $\gamma$-cover. Also, every infinite subset of a $\gamma$-cover is itself a $\gamma$-cover. These facts will be used without special mention. For a space $X$ $C_p(X)$ denotes the space of all continuous real-valued functions on $X$ endowed with the pointwise topology. Since $C_p(X)$ is a homogeneous space we can single out the point $0 \in C_p(X)$, the function which has value zero everywhere. For $Y \subset X$, $\pi$ denotes the mapping from $C_p(X)$ into $C_p(Y)$ defined by $\pi(f) = f|_Y$, $f \in C_p(X)$.

Some results in the literature show that there is a duality between relative covering properties of a subspace $Y$ of a Tychonoff space $X$ and closure-type properties of the mapping $\pi$. This sort of duality was documented, for example, for the Lindelöf property [G], the Menger property (and variations of this property) [KB], [GK], Rothberger’s property [KB] and the Hurewicz property [GK].

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In this paper we give such a duality analysis concerning relative $\gamma$-set property and the corresponding strongly Fréchet property of the mapping $\pi$.

2. Results

In [GN], Gerlits and Nagy introduced the following notion: a space $X$ is a $\gamma$-space (or a $\gamma$-set) if for each $\omega$-cover $U$ of $X$ one can choose a sequence $(U_n : n \in \mathbb{N})$ such that for each $n$ $U_n \in U$ and $\{U_n : n \in \mathbb{N}\}$ is a $\gamma$-cover of $X$. It is known that the $\gamma$-set property of a space $X$ is equivalent to the statement: For each sequence $(U_n : n \in \mathbb{N})$ of $\omega$-covers of $X$ there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n$ $\mathcal{V}_n$ is a finite subset of $U_n$ and $\bigcup_{n \in \mathbb{N}} \mathcal{V}_n$ is a $\gamma$-cover of $X$ (see, for example, [JMSS]). It is easy now to conclude that a space $X$ is actually a $\gamma$-set if and only if for each sequence $(U_n : n \in \mathbb{N})$ of $\omega$-covers of $X$ there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n$ $\mathcal{V}_n$ is a finite subset of $U_n$ and for each finite set $F \subset X$, for all but finitely many $n$, $F \subset \bigcap \mathcal{V}_n$.

We introduce a relative version of this notion.

2.1. Definition. Let $Y$ be a subspace of a space $X$. We say that $Y$ is a $\gamma$-set in $X$ if for each sequence $(U_n : n \in \mathbb{N})$ of $\omega$-covers of $X$ there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n$ $\mathcal{V}_n$ is a finite subset of $U_n$ and for each finite set $F \subset Y$, for all but finitely many $n$, $F \subset \bigcap \mathcal{V}_n$. □

Remark. (1) Let us note that the relative $\gamma$-set property is hereditary. However, the Gerlits-Nagy $\gamma$-set property is not hereditary. In [GM], Galvin and Miller, under appropriate hypotheses, constructed an uncountable set $X$ of the real line which is a $\gamma$-set and is concentrated [Mi] on some countable set $C \subset X$. It follows that $X \setminus C$ is not a $\gamma$-set.

(2) The same set $X$ is a relative $\gamma$-set (because, clearly, every $\gamma$-set is also a relative $\gamma$-set). Thus $X \setminus C$ is a relative $\gamma$-set which is not a $\gamma$-set.

(3) Relative $\gamma$-sets of real numbers have Borel’s property of strong measure zero [B]. Thus, by well known results about strong measure zero sets (Borel’s conjecture that no uncountable set of real numbers has strong measure zero is undecidable in ZFC) [Mi], the question if there is an uncountable relative $\gamma$-set of real numbers is undecidable in ZFC. □

2.2. Lemma. ([JMSS]) If $U$ is an $\omega$-cover of a space $X^2$, then there is an $\omega$-cover $V$ of $X$ such that $\{V^2 : V \in V\}$ refines $U$. □

2.3. Theorem. For a space $X$ and $Y \subset X$ the following are equivalent:

(1) $Y$ is a $\gamma$-set in $X$;

(2) $Y^2$ is a $\gamma$-set in $X^2$ (and thus for each positive integer $n$, $Y^n$ is a $\gamma$-set in $X^n$).
Proof. (1) \(\Rightarrow\) (2): Let \((U_n : n \in \mathbb{N})\) be a sequence of \(\omega\)-covers of \(X^2\). For each \(n \in \mathbb{N}\) let, by Lemma 2.2, \(V_n\) be an \(\omega\)-cover of \(X\) such that \(\{V^2 : V \in V_n\}\) refines \(U_n\). Since \(Y\) is a \(\gamma\)-set in \(X\) one can find a sequence \((W_n : n \in \mathbb{N})\) of finite sets such that for each \(n\) \(W_n \subset V_n\) and for each finite set \(K \subset Y\) there is \(n_0\) such that \(F \subset W\) for each \(W \in W_n\) with \(n > n_0\). For each \(n\) we let \(C_n\) denote the subset of \(U_n\) satisfying: for each \(W \in W_n\) there exists \(U \in C_n\) for which \(W^2 \subset U\). We claim that \((C_n : n \in \mathbb{N})\) witnesses that \(Y^2\) is a \(\gamma\)-set in \(X^2\).

Let \(P = \{p_1, p_2, \ldots, p_k\}\) be a finite subset of \(Y^2\) and let for each \(i = 1, \ldots, k\), \(p_i = (p_{i1}, p_{i2})\). Then \(Q = \{p_{ij} : i = 1, \ldots, k; j = 1, 2\}\) is a finite subset of \(Y\), so that there is \(m\) such that \(Q \subset W\) for each \(W \in W_n\), \(n > m\). For each such \(W\) take a \(U \in C_n\) for which \(W^2 \subset U\). Then for each \(U \in C_n\) and each \(n > m\) we have \(P \subset Q^2 \subset W^2 \subset U\), i.e. \(Y^2\) is a \(\gamma\)-set in \(X^2\).

(2) \(\Rightarrow\) (1): Let \((U_n : n \in \mathbb{N})\) be a sequence of \(\omega\)-covers of \(X\). For each \(n\) let \(V_n = \{U^2 : U \in U_n\}\). It is easy to see that \((V_n : n \in \mathbb{N})\) is a sequence of \(\omega\)-covers of \(X^2\). By assumption, for every \(n \in \mathbb{N}\) we can choose a finite \(W_n \subset U_n\) such that for each finite subset \(K\) of \(Y^2\) there exists \(n_0\) such that \(K \subset U^2\) for each \(U \in W_n\) and each \(n > n_0\). We verify that the sequence \((W_n : n \in \mathbb{N})\) witnesses for \((U_n : n \in \mathbb{N})\) that \(Y\) is a \(\gamma\)-set in \(X\).

Let \(S\) be a finite subset of \(Y\). Since \(S^2\) is a finite subset of \(Y^2\) there is some \(k\) such that \(S^2 \subset U^2\) for all \(U \in W_n\) with \(n > k\). It implies \(S \subset U\) for each \(U \in W_n\), \(n > k\). \(\square\)

For the main result of the paper we need the following definition.

2.4. Definition. A continuous mapping \(f : X \to Y\) is said to be strongly Fréchet if for each sequence \((A_n : n \in \mathbb{N})\) of subsets of \(X\) and each \(x \in \bigcap_{n \in \mathbb{N}} A_n\) there is a sequence \((B_n : n \in \mathbb{N})\) such that for each \(n\) \(B_n\) is a finite, nonempty subset of \(A_n\) and for each neighborhood \(V\) of \(f(x)\) there exists \(n_0 \in \mathbb{N}\) such that \(f(B_n) \subset V\) for all \(n > n_0\) (i.e. the sequence \((f(B_n) : n \in \mathbb{N})\) converges to \(f(x)\)). \(\square\)

Let us note that if either \(X\) or \(Y\) is strongly Fréchet, then \(f\) is strongly Fréchet.

In [GN], it was shown that a space \(X\) is a \(\gamma\)-set if and only if the space \(C_p(X)\) is strongly Fréchet. We shall show now that a similar assertion is true for relative \(\gamma\)-property and the corresponding mapping between function spaces.

2.5. Theorem. For a Tychonoff space \(X\) and its subspace \(Y\) the following are equivalent:

(a) \(Y\) is a \(\gamma\)-set in \(X\);
(a') For each \(n \in \mathbb{N}\), \(Y^n\) is a \(\gamma\)-set in \(X^n\);
(b) The mapping $\pi$ is strongly Fréchet.

Proof. (a) $\implies$ (b): Let $(A_n : n \in \mathbb{N})$ be a sequence of subsets of $C_p(X)$ such that $\emptyset \in \bigcap_{n \in \mathbb{N}} A_n$. Fix $n$. For every finite subset $F$ of $X$ the basic neighborhood $W = W(0; F; 1/n)$ of $0$ intersects $A_n$; pick a function $f_{F,n} \in A_n$ such that $|f_{F,n}(x)| < 1/n$ for each $x \in F$. Since $f_{F,n}$ is a continuous function there are neighborhoods $U_x$ of $x$, $x \in F$, such that for $U_{F,n} = \bigcup_{x \in F} U_x \supset F$ it holds $f_{F,n}(U_{F,n}) \subset (-1/n, 1/n)$. If $U_n = \{ U_{F,n} : F \in \mathcal{F} \}$, then $(U_n : n \in \mathbb{N})$ is a sequence of $\omega$-covers of $X$. By assumption one can find a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and each finite subset of $Y$ is contained in all elements of $\mathcal{V}_k$ for all $k$ bigger than some $n_0 \in \mathbb{N}$. Let (for each $n$) $\mathcal{V}_n = \{ U_{F,n} : i \in M_n \}$, where $M_n$ is a finite subset of $\mathbb{N}$. Consider the sets $B_n$, $n \in \mathbb{N}$, of the corresponding functions: $B_n = \{ f_{F,n} : i \in M_n \} \subset A_n$. We verify that the sequence $(B_n : n \in \mathbb{N})$ witnesses for $(A_n : n \in \mathbb{N})$ that $\pi$ is strongly Fréchet. Let $W = W(\pi(0); K; \varepsilon)$ be a neighborhood of $\pi(0)$ in $C_p(Y)$ and suppose that $m$ is a positive integer such that $1/m < \varepsilon$. Since $K$ is a finite subset of $Y$ and $Y$ is a $\gamma$-set in $X$ there is $n_0 \in \mathbb{N}$ such that $K \subset \bigcap \mathcal{V}_k$ for each $k > n_0$. This means that for each $i \in M_k$, $k > n_0$, it holds $\pi(f_{F,k})(K) \subset (-1/k, 1/k)$. For all $n > \max\{n_0, m\}$ and all $i \in M_n$ we have

$$\pi(f_{F,n})(K) = f_{F,n}(K) \subset f_{F,n}(U_{F,n}) \subset (-1/n, 1/n) \subset (-\varepsilon, \varepsilon),$$

i.e. $\pi(B_n) \subset W$ for each $n > \max\{n_0, m\}$.

(b) $\implies$ (a): Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of $\omega$-covers of $X$. For each $n \in \mathbb{N}$ and each finite subset $F$ of $X$ we denote by $\mathcal{U}_{n,F}$ the set $\{ U \in \mathcal{U}_n : F \subset U \}$. If $U \in \mathcal{U}_{n,F}$, let $f_{U,F} : X \to [0, 1]$ be a continuous function satisfying $f_{U,F}(F) = 0$, $f_{U,F}(X \setminus U) = 1$. Let $A_n = \{ f_{U,F} : F \in \mathcal{F}, U \in \mathcal{U}_{n,F} \}$. Then $0 \in \bigcap_{n \in \mathbb{N}} A_n$: if $W(\pi(0); K; \varepsilon)$ is a basic neighborhood of $0$ and $U \in \mathcal{U}_{n,K}$, then the function $f_{U,K}$ belongs to $A_n \bigcap W(\pi(0); K; \varepsilon)$ for each $n$.

Since $\pi$ is strongly Fréchet there exists a sequence $(B_n : n \in \mathbb{N})$ of finite sets such that for each $n$, $B_n \subset A_n$ and $(\pi(B_n) : n \in \mathbb{N})$ converges to $\pi(0)$. Assume that for $n \in \mathbb{N}$, $B_n = \{ f_{U_i,F} : i \in Z_n \}$, where $Z_n$ is a finite subset of $\mathbb{N}$. Consider $\mathcal{V}_n = \{ U_i : i \in Z_n \} \subset \mathcal{U}_n$, and prove that the sequence $(\mathcal{V}_n : n \in \mathbb{N})$ witnesses for $(\mathcal{U}_n : n \in \mathbb{N})$ that $Y$ is a $\gamma$-set in $X$. Let $S$ be a finite subset of $Y$. Then there exists $n_0 \in \mathbb{N}$ such that the neighborhood $W = W(\pi(0); S; 1)$ of $\pi(0) \in C_p(Y)$ contains all $\pi(B_n)$ with $n > n_0$, i.e. $\pi(f_{U_i,F}) \in W$ for each $i \in Z_n$, $n > n_0$. This implies $S \subset U_i$ for each $i \in Z_n$, $n > n_0$, i.e. (a) is satisfied. \hfill $\square$

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