ON RELATIVE γ-SETS

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Abstract. In this note we show a relative version of the Gerlits-Nagy theorem concerning a characterization of γ-sets in terms of function spaces. Our result involves a property of the corresponding mapping between function spaces.

1. Introduction

In this note X will denote a Tychonoff space and Y will be a subspace of X. The notation and terminology we follow are standard [E]. An open cover $\mathcal{U}$ of a space X is an *ω-cover* [GN] if X does not belong to $\mathcal{U}$ and every finite subset of X is contained in an element of $\mathcal{U}$. An open cover $\mathcal{U}$ of X is called a *γ-cover* [GN] if it is infinite and for each $x \in X$ the set $\{U \in \mathcal{U} : x \notin U\}$ is finite. Let us observe that every γ-cover is an ω-cover; moreover, each finite subset of a space belongs to all but finitely many elements of a γ-cover. Also, every infinite subset of a γ-cover is itself a γ-cover. These facts will be used without special mention. For a space X $C_p(X)$ denotes the space of all continuous real-valued functions on X endowed with the pointwise topology. Since $C_p(X)$ is a homogeneous space we can single out the point $\underline{0} \in C_p(X)$, the function which has value zero everywhere. For $Y \subset X$, π denotes the mapping from $C_p(X)$ into $C_p(Y)$ defined by $\pi(f) = f \upharpoonright Y$, $f \in C_p(X)$.

Some results in the literature show that there is a duality between relative covering properties of a subspace Y of a Tychonoff space X and closure-type properties of the mapping π. This sort of duality was documented, for example, for the Lindelöf property [G], the Menger property (and variations of this property) [KB], [GK], Rothberger's property [KB] and the Hurewicz property [GK].

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In this paper we give such a duality analysis concerning relative γ-set property and the corresponding strongly Fréchet property of the mapping π.

2. Results

In [GN], Gerlits and Nagy introduced the following notion: a space X is a *γ-space* (or a *γ-set*) if for each ω-cover $\mathcal{U}$ of X one can choose a sequence $(U_n : n \in \mathbb{N})$ such that for each n $U_n \in \mathcal{U}$ and $\{U_n : n \in \mathbb{N}\}$ is a γ-cover of X. It is known that the γ-set property of a space X is equivalent to the statement: For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of ω-covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each n $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is a γ-cover of X (see, for example, [JMSS]). It is easy now to conclude that a space X is actually a γ-set if and only if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of ω-covers of X there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each n $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each finite set $F \subset X$, for all but finitely many n, $F \subset V$ for each $V \in \mathcal{V}_n$.

We introduce a relative version of this notion.

2.1. Definition. Let Y be a subspace of a space X. We say that Y is a *γ-set in X* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of ω-covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each n $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each finite set $F \subset Y$, for all but finitely many n, $F \subset \cap\mathcal{V}_n$. □

Remark. (1) Let us note that the relative γ-set property is hereditary. However, the Gerlits-Nagy γ-set property is not hereditary. In [GM], Galvin and Miller, under appropriate hypotheses, constructed an uncountable set X of the real line which is a γ-set and is concentrated [Mi] on some countable set $C \subset X$. It follows that $X \setminus C$ is not a γ-set.

(2) The same set X is a relative γ-set (because, clearly, every γ-set is also a relative γ-set). Thus $X \setminus C$ is a relative γ-set which is not a γ-set.

(3) Relative γ-sets of real numbers have Borel's property of strong measure zero [B]. Thus, by well known results about strong measure zero sets (Borel's conjecture that no uncountable set of real numbers has strong measure zero is undecidable in ZFC) [Mi], the question if there is an uncountable relative γ-set of real numbers is undecidable in ZFC. □

2.2. Lemma. ([JMSS]) *If $\mathcal{U}$ is an ω-cover of a space X^2, then there is an ω-cover $\mathcal{V}$ of X such that $\{V^2 : V \in \mathcal{V}\}$ refines $\mathcal{U}$.* □

2.3. Theorem. *For a space X and $Y \subset X$ the following are equivalent:*

(1) *Y is a γ-set in X;*

(2) *Y^2 is a γ-set in X^2 (and thus for each positive integer n, Y^n is a γ-set in X^n).*

Proof. (1) $\implies$ (2): Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of ω-covers of X^2. For each $n \in \mathbb{N}$ let, by Lemma 2.2, $\mathcal{V}_n$ be an ω-cover of X such that $\{V^2 : V \in \mathcal{V}_n\}$ refines $\mathcal{U}_n$. Since Y is a γ-set in X one can find a sequence $(\mathcal{W}_n : n \in \mathbb{N})$ of finite sets such that for each n $\mathcal{W}_n \subset \mathcal{V}_n$ and for each finite set $K \subset Y$ there is n_0 such that $F \subset W$ for each $W \in \mathcal{W}_n$ with $n > n_0$. For each n we let $\mathcal{C}_n$ denote the subset of $\mathcal{U}_n$ satisfying: for each $W \in \mathcal{W}_n$ there exists $U \in \mathcal{C}_n$ for which $W^2 \subset U$ holds. We claim that $(\mathcal{C}_n : n \in \mathbb{N})$ witnesses that Y^2 is a γ-set in X^2.

Let $P = \{p_1, p_2, \ldots, p_k\}$ be a finite subset of Y^2 and let for each $i = 1, \ldots, k$, $p_i = (p_{i1}, p_{i2})$. Then $Q = \{p_{ij} : i = 1, \ldots, k;\ j = 1, 2\}$ is a finite subset of Y, so that there is m such that $Q \subset W$ for each $W \in \mathcal{W}_n$, $n > m$. For each such W take a $U \in \mathcal{C}_n$ for which $W^2 \subset U$. Then for each $U \in \mathcal{C}_n$ and each $n > m$ we have $P \subset Q^2 \subset W^2 \subset U$, i.e. Y^2 is a γ-set in X^2.

(2) $\implies$ (1): Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of ω-covers of X. For each n let $\mathcal{V}_n = \{U^2 : U \in \mathcal{U}_n\}$. It is easy to see that $(\mathcal{V}_n : n \in \mathbb{N})$ is a sequence of ω-covers of X^2. By assumption, for every $n \in \mathbb{N}$ we can choose a finite $\mathcal{W}_n \subset \mathcal{U}_n$ such that for each finite subset K of Y^2 there exists n_0 such that $K \subset U^2$ for each $U \in \mathcal{W}_n$ and each $n > n_0$. We verify that the sequence $(\mathcal{W}_n : n \in \mathbb{N})$ witnesses for $(\mathcal{U}_n : n \in \mathbb{N})$ that Y is a γ-set in X.

Let S be a finite subset of Y. Since S^2 is a finite subset of Y^2 there is some k such that $S^2 \subset U^2$ for all $U \in \mathcal{W}_n$ with $n > k$. It implies $S \subset U$ for each $U \in \mathcal{W}_n$, $n > k$. $\square$

For the main result of the paper we need the following definition.

2.4. Definition. A continuous mapping $f : X \to Y$ is said to be *strongly Fréchet* if for each sequence $(A_n : n \in \mathbb{N})$ of subsets of X and each $x \in \bigcap_{n \in \mathbb{N}} \overline{A_n}$ there is a sequence $(B_n : n \in \mathbb{N})$ such that for each n B_n is a finite, nonempty subset of A_n and for each neighborhood V of $f(x)$ there exists $n_0 \in \mathbb{N}$ such that $f(B_n) \subset V$ for all $n > n_0$ (i.e. the sequence $(f(B_n) : n \in \mathbb{N})$ converges to $f(x)$). $\square$

Let us note that if either X or Y is strongly Fréchet, then f is strongly Fréchet.

In [GN], it was shown that a space X is a γ-set if and only if the space $C_p(X)$ is strongly Fréchet. We shall show now that a similar assertion is true for relative γ-property and the corresponding mapping between function spaces.

2.5. Theorem. *For a Tychonoff space X and its subspace Y the following are equivalent:*

(*a*) *Y is a γ-set in X;*

(*a′*) *For each $n \in \mathbb{N}$, Y^n is a γ-set in X^n;*

(*b*) *The mapping* π *is strongly Fréchet.*

Proof. $(a) \implies (b)$: Let $(A_n : n \in \mathbb{N})$ be a sequence of subsets of $C_p(X)$ such that $\underline{0} \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$. Fix n. For every finite subset F of X the basic neighborhood $W = W(\underline{0}; F; 1/n)$ of $\underline{0}$ intersects A_n; pick a function $f_{F,n} \in A_n$ such that $|f_{F,n}(x)| < 1/n$ for each $x \in F$. Since $f_{F,n}$ is a continuous function there are neighborhoods U_x of x, $x \in F$, such that for $U_{F,n} = \bigcup_{x\in F} U_x \supset F$ it holds $f_{F,n}(U_{F,n}) \subset (-1/n, 1/n)$. If $\mathcal{U}_n = \{U_{F,n} : F \in [X]^{<\omega}\}$, then $(\mathcal{U}_n : n \in \mathbb{N})$ is a sequence of ω-covers of X. By assumption one can find a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each n $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and each finite subset of Y is contained in all elements of $\mathcal{V}_k$ for all k bigger than some $n_0 \in \mathbb{N}$. Let (for each n) $\mathcal{V}_n = \{U_{F_i,n} : i \in M_n\}$, where M_n is a finite subset of $\mathbb{N}$. Consider the sets B_n, $n \in \mathbb{N}$, of the corresponding functions: $B_n = \{f_{F_i,n} : i \in M_n\} \subset A_n$. We verify that the sequence $(B_n : n \in \mathbb{N})$ witnesses for $(A_n : n \in \mathbb{N})$ that π is strongly Fréchet.

Let $W = W(\pi(\underline{0}); K; \varepsilon)$ be a neighborhood of $\pi(\underline{0})$ in $C_p(Y)$ and suppose that m is a positive integer such that $1/m < \varepsilon$. Since K is a finite subset of Y and Y is a γ-set in X there is $n_0 \in \mathbb{N}$ such that $K \subset \bigcap \mathcal{V}_k$ for each $k > n_0$. This means that for each $i \in M_k$, $k > n_0$, it holds $\pi(f_{F_i,k})(K) \subset (-1/k, 1/k)$. For all $n > \max\{n_0, m\}$ and all $i \in M_n$ we have

$$\pi(f_{F_i,n})(K) = f_{F_i,n}(K) \subset f_{F_i,n}(U_{F_i,n}) \subset (-1/n, 1/n) \subset (-\varepsilon, \varepsilon),$$

i.e. $\pi(B_n) \subset W$ for each $n > \max\{n_0, m\}$.

$(b) \implies (a)$: Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of ω-covers of X. For each $n \in \mathbb{N}$ and each finite subset F of X we denote by $\mathcal{U}_{n,F}$ the set $\{U \in \mathcal{U}_n : F \subset U\}$. If $U \in \mathcal{U}_{n,F}$, let $f_{U,F} : X \to [0,1]$ be a continuous function satisfying $f_{U,F}(F) = 0$, $f_{U,F}(X \setminus U) = 1$. Let $A_n = \{f_{U,F} : F \in [X]^{<\omega}, U \in \mathcal{U}_{n,F}\}$. Then $\underline{0} \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$: if $W(\underline{0}; K; \varepsilon)$ is a basic neighborhood of $\underline{0}$ and $U \in \mathcal{U}_{n,K}$, then the function $f_{U,K}$ belongs to $A_n \bigcap W(\underline{0}; K; \varepsilon)$ for each n.

Since π is strongly Fréchet there exists a sequence $(B_n : n \in \mathbb{N})$ of finite sets such that for each n, $B_n \subset A_n$ and $(\pi(B_n) : n \in \mathbb{N})$ converges to $\pi(\underline{0})$. Assume that for $n \in \mathbb{N}$, $B_n = \{f_{U_i,F_i} : i \in Z_n\}$, where Z_n is a finite subset of $\mathbb{N}$. Consider $\mathcal{V}_n = \{U_i : i \in Z_n\} \subset \mathcal{U}_n$ and prove that the sequence $(\mathcal{V}_n : n \in \mathbb{N})$ witnesses for $(\mathcal{U}_n : n \in \mathbb{N})$ that Y is a γ-set in X. Let S be a finite subset of Y. Then there exists $n_0 \in \mathbb{N}$ such that the neighborhood $W = W(\pi(\underline{0}); S; 1)$ of $\pi(\underline{0}) \in C_p(Y)$ contains all $\pi(B_n)$ with $n > n_0$, i.e. $\pi(f_{U_i,F_i}) \in W$ for each $i \in Z_n$, $n > n_0$. This implies $S \subset U_i$ for each $i \in Z_n$, $n > n_0$, i.e. (*a*) is satisfied. □

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