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Topology and its Applications 154 (2007) 1971–1979
Topology and its Applications
www.elsevier.com/locate/topol

When does the Haver property imply selective screenability?

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Received 26 September 2006; accepted 12 February 2007

Abstract

We point out that in metric spaces Haver's property is not equivalent to the property introduced by Addis and Gresham. We prove that they are equal when the space has the Hurewicz property. We prove several results about the preservation of Haver's property in products. We show that if a separable metric space has the Haver property, and the nth power has the Hurewicz property, then the nth power has the Addis–Gresham property. R. Pol showed earlier that this is not the case when the Hurewicz property is replaced by the weaker Menger property. We introduce new classes of weakly infinite dimensional spaces.

MSC: primary 54D20, 54D45, 55M10; secondary 03E20

Keywords: Haver property; Selective screenability; Hurewicz property; Menger property; Strongly countable dimensional; Countable dimensional; Weakly infinite dimensional; Selection principle

In [6] Haver introduced for metric space (X, d) the following property: There is for each sequence $(\varepsilon_n\colon n < \infty)$ of positive real numbers a corresponding sequence $(\mathcal{V}_n\colon n < \infty)$ where each $\mathcal{V}_n$ is a pairwise disjoint family of open sets, each of diameter less than ε_n, such that $\bigcup_{n<\infty} \mathcal{V}_n$ is a cover of X. When a metric space has this property we say it has the *Haver property*. We consider the Haver property's relation to selection principles.

Let $\mathcal{A}$ and $\mathcal{B}$ be given families of collections of subsets of some set S. Then the following symbols and statements define selection principles for the pair $\mathcal{A}, \mathcal{B}$:

- $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$: For each sequence $(O_m\colon m < \infty)$ of elements of $\mathcal{A}$ there is a sequence $(T_m\colon m < \infty)$ with each $T_m \in O_m$, and $\{T_m\colon m < \infty\} \in \mathcal{B}$.
- $\mathsf{S}_{\mathrm{fin}}(\mathcal{A}, \mathcal{B})$: For each sequence $(O_m\colon m < \infty)$ of elements of $\mathcal{A}$ there is a sequence $(T_m\colon m < \infty)$ with each T_m a finite subset of O_m, and $\bigcup\{T_m\colon m < \infty\} \in \mathcal{B}$.
- $\mathsf{S}_c(\mathcal{A}, \mathcal{B})$: For each sequence $(O_m\colon m < \infty)$ of elements of $\mathcal{A}$ there is a sequence $(T_m\colon m < \infty)$ with each T_m a pairwise disjoint family refining O_m, and $\bigcup\{T_m\colon m < \infty\} \in \mathcal{B}$.

It is clear that $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ implies each of $\mathsf{S}_{\mathrm{fin}}(\mathcal{A}, \mathcal{B})$ and $\mathsf{S}_c(\mathcal{A}, \mathcal{B})$. Even for very standard examples of $\mathcal{A}$ and $\mathcal{B}$ no other implications hold. For example, let S be a topological space, and let $\mathcal{O}$ denote the collection of open covers

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doi:10.1016/j.topol.2007.02.004

of S. If we take $\mathcal{A} = \mathcal{O} = \mathcal{B}$, then $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ is the Rothberger property introduced in [14], $\mathsf{S}_{\mathrm{fin}}(\mathcal{A}, \mathcal{B})$ is the Menger property introduced by W. Hurewicz in [7], and $\mathsf{S}_c(\mathcal{A}, \mathcal{B})$ is the property C introduced in [1] by Addis and Gresham. In metrizable spaces, $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ implies the space is zero-dimensional. Recall that a space is said to be *countable dimensional* if it is a union of countably many zero-dimensional subsets. It is also well known (Corollary 2.10 of [1]) that countable dimensionality implies $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$. In [12] the author proved that infinite dimensionality does not imply property C. Thus the Hilbert cube $[0, 1]^{\mathbb{N}}$ does not have $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ and $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$, but it is compact and so has the Menger property $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O})$. The subspace $\{\frac{1}{n}\colon n \in \mathbb{N}\}^{\mathbb{N}}$ of the Hilbert cube is homeomorphic to the set of irrational numbers and thus does not have the Menger property $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O})$, but it is zero-dimensional and thus has the property $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$.

Since $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$ is a selective version of Bing's screenability property (p. 176 of [4]), we call it *selective screenability*.

Summary of results

In metrizable spaces $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$ implies the Haver property. But the converse implication does not hold: In [12] R. Pol constructed a compact metrizable space $X = M \cup L$ such that the subspace M does not have property $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$, but X does have $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$. Since the Haver property is inherited by subspaces, the subspace M of X has the Haver property, but not $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$. So when does a metric space with the Haver property have $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$? It is easy to see that σ-compact metric spaces with the Haver property have $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$. We shall show that σ-compactness can be weakened considerably to a selection principle called the Hurewicz property:

Theorem 1. *Let (X, d) be a metrizable space with the Hurewicz property. If X has the Haver property, then it has property $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$.*

The Hurewicz property will be defined below. In his paper [13] Rohm asked when the product of two spaces with property $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$ again has this property. In [5] and [13] the authors prove the following:

Theorem 2 *(Hattori, Yamada and Rohm). Let X and Y be topological spaces satisfying $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$. If X is σ-compact, then $X \times Y$ has the property $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$.*

To what extent is it necessary to assume that X is σ-compact? R. Pol considered this question in [11]. It is well known that σ-compactness implies the Menger property $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O})$, but is not equivalent to it. R. Pol showed that σ-compactness cannot be weakened to $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O})$:

Theorem 3 *(R. Pol). Assume the Continuum Hypothesis. Then there is for each positive integer n a separable metric space X such that X^n has $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O}) + \mathsf{S}_c(\mathcal{O}, \mathcal{O})$, and X^{n+1} has $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O}) + \neg\mathsf{S}_c(\mathcal{O}, \mathcal{O})$.*

And E. Pol examined to what extent the property $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$ must be strengthened if we drop the assumption of σ-compactness. A first strengthening would be to assume zero-dimensionality. She showed that this of itself is not enough and obtained the following examples in [9] and [10]:

Theorem 4 *(E. Pol).*

(1) *There is a separable metric space X such that for each n, X^n satisfies $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$, but there is a subset Y of the irrational numbers such that $X \times Y$ does not satisfy $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$.*
(2) *Assume the Continuum Hypothesis. Then there is a separable metric space X such that X has property $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$, but its product with the irrational numbers does not.*

It is well known that σ-compactness implies the Hurewicz covering property which in turn implies the Menger property $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O})$, and that these three properties are mutually inequivalent. We prove a theorem, Theorem 8, which implies:

Corollary 9. *Let X and Y be metric spaces with the Haver property. If X has the Hurewicz property, then $X \times Y$ has the Haver property.*

In particular, we prove Theorem 12 which implies the following finite powers result for metrizable spaces:

Corollary 13. *If X is a metric space and n a positive integer such that X satisfies $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$ and X^n satisfies the Hurewicz property, then X^n also satisfies $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$.*

R. Pol's examples in Theorem 3 shows that the Hurewicz property in Corollary 13 cannot be weakened to the Menger property.

Theorem 15. *Let X and Y be metric spaces. If X has the Haver property and Y is countable dimensional then $X \times Y$ has the Haver property.*

We also obtain a third product theorem for the Haver property:

Theorem 16. *Let X be a complete metric space with the Haver property. For each metric space Y with the Haver property, $X \times Y$ has the Haver property.*

Theorems 15 and 16 are applied to show that the examples constructed by E. Pol for product failures for $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$ do not witness the corresponding product failures for the Haver property.

We consider the selective screenability property for the classes of open covers of X introduced in [3]: Let FD denote the collection of finite dimensional subsets of X. Then $\mathcal{O}_{\mathsf{fd}}$ denotes the open covers $\mathcal{U}$ of X such that $X \notin \mathcal{U}$, but for each $C \in \mathsf{FD}$ there is a $U \in \mathcal{U}$ with $C \subseteq U$. We show that for metrizable space X, $\mathsf{S}_1(\mathcal{O}_{\mathsf{fd}}, \mathcal{O})$ implies $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$, but the converse is not true. We show that $\mathsf{S}_1(\mathcal{O}_{\mathsf{cfd}}, \mathcal{O})$ defines a new class of weakly infinite dimensional spaces.

The last section states some specific problems suggested by our results.

1. The Haver property and the Hurewicz property

In this section we prove Theorem 1. In fact, this theorem follows directly from Theorem 5 below. A topological space X has the *Hurewicz property* if there is for each sequence $(\mathcal{U}_n\colon n < \infty)$ of open covers of X a sequence $(\mathcal{V}_n\colon n < \infty)$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, and each element of X is in all but finitely many of the sets $\bigcup \mathcal{V}_n$. Several alternative useful characterizations of the Hurewicz property are known.

Let X be a metrizable space and let d be a metric such that (X, d) has the Haver property. For an $x \in X$ and a positive real number r, $B(x, r)$ denotes the set $\{y \in X\colon d(x, y) < r\}$. Also assume that X has the Hurewicz property.

Now let X be a subspace of the metric space Y. The symbol $\mathcal{O}_{YX}$ denotes the family of covers of X by sets open in Y. In [2] we considered the relative version of $\mathsf{S}_c(\mathcal{A}, \mathcal{B})$ for various types of covers $\mathcal{A}$ and $\mathcal{B}$ of topological spaces. We now consider $\mathsf{S}_c(\mathcal{O}_Y, \mathcal{O}_{XY})$ for separable metrizable spaces. The following fact is useful. Let d be a metric on Y. If $\varepsilon > 0$ is given and if $S \subset X$ is a set of d-diameter less than ε, then there is an open set $U \subset Y$ such that $S \subseteq U$ and the d-diameter of U is less than ε.

Theorem 5. *Let Y be a metrizable space which has the Hurewicz property. Then for a subspace X of Y the following are equivalent*:

(1) *$\mathsf{S}_c(\mathcal{O}_Y, \mathcal{O}_{YX})$ holds.*
(2) *X has the Haver property in all equivalent metrics on Y.*
(3) *X has the Haver property in some equivalent metric on Y.*

Proof. The proofs of (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are easy.

(3) $\Rightarrow$ (1) Let $(\mathcal{U}_n\colon n < \infty)$ be a sequence of open covers of Y. Let d be an equivalent metric on Y in which Y has the Haver property. Define for each n a new open cover $\mathcal{H}_n$ as follows:

$$\mathcal{H}_n := \bigl\{B(y, \varepsilon)\colon \varepsilon > 0,\ y \in Y \text{ and } (\exists U \in \mathcal{U}_n)\bigl(B(y, 3 \cdot \varepsilon) \subset U\bigr)\bigr\}.$$

Applying the Hurewicz property on Y, choose for each n a finite set $\mathcal{F}_n \subset \mathcal{H}_n$ such that each element of Y is in all but finitely many $\bigcup \mathcal{F}_n$'s.

For each n: Representing $\mathcal{F}_n$ as $\{B(y_i^n, \varepsilon_i^n)\colon i \in I_n\}$, where each I_n is some finite set, we define $\varepsilon_n = \min\{\varepsilon_i^n\colon i \in I_n\}$. Each ε_n is positive. Also, represent $\mathbb{N}$, the set of natural numbers, as $\mathbb{N} = \bigcup_{k<\infty} J_k$ where each J_k is infinite and for $k \neq \ell$, $J_k \cap J_\ell = \emptyset$.

Applying the Haver property of X for the metric d on Y to each of the sequences $(\varepsilon_n\colon n \in Y_k)$, we find for each k a sequence $(\mathcal{S}_n\colon n \in J_k)$ such that each $\mathcal{S}_n$ is a disjoint family of subsets of X open in X, each of diameter less than ε_n, and $\bigcup_{n \in J_k} \mathcal{S}_n$ is an open (in the relative topology of X) cover of X. By Theorem II.21.XI.1 (p. 226 of [8]) choose for each $n \in J_k$ and each $V \in \mathcal{S}_n$ an open subset T_V of Y such that $V = X \cap T_V$, and when $V \neq U$ are elements of $\mathcal{S}_n$, then $T_U \cap T_V = \emptyset$. By the remark preceding Theorem 5, we may assume that $\mathrm{diam}_d(T_V)$ is less than ε_n. Then put $\mathcal{T}_n = \{T_V\colon V \in \mathcal{S}_n\}$.

Now we define refinements of the $\mathcal{U}_n$'s: For each n define $\mathcal{V}_n := \{T_V \in \mathcal{T}_n\colon (\exists U \in \mathcal{U}_n)(T_V \subseteq U)\}$. Observe that $\mathcal{V}_n$ is pairwise disjoint and refines $\mathcal{U}_n$. We show that $\bigcup_{n<\infty} \mathcal{V}_n$ is a cover of X.

Consider any $x \in X$. Fix N_x so large that for all $n \geqslant N_x$ we have $x \in \bigcup \mathcal{F}_n$. Then fix a k so large that $N_x < \min(J_k)$. Consider any $m \in J_k$ with $x \in \bigcup \mathcal{S}_m$, and choose a $J \in \mathcal{S}_m$ with $x \in J$. By choice of k we also have $x \in \bigcup \mathcal{F}_m$. Choose an $i \in I_m$ with $x \in B(x_i^m, \varepsilon_i^m)$. From the definition of $\mathcal{H}_m$ fix a $G \in \mathcal{U}_m$ such that $B(x_i^m, 3 \cdot \varepsilon_i^m) \subset G$. Consider any $y \in T_J$. Then we have $d(y, x_i^m) \leqslant d(x, y) + d(x, x_i^m) < 3 \cdot \varepsilon_i^m$, and so $y \in B(x_i^m, 3 \cdot \varepsilon_i^m) \subset G$ and we have $T_J \in \mathcal{V}_m$. Thus, we found an m and an element of $\mathcal{V}_m$ which contains x. □

2. Products and the Haver property

Next we investigate Rohm's question regarding products of $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$-spaces for the Haver property instead: When is the product of two metric spaces with the Haver property a space with the Haver property?

An open cover of a space is said to be *large* if each element of X belongs to infinitely many different elements of the cover. Recall from [15] that a space X has *the grouping property* if there is for each bijectively enumerated large cover $(U_n\colon n < \infty)$ a sequence $n_n < n_2 < \cdots < n_k < \cdots$ such that each element of X is in all but finitely many of the sets $\bigcup_{n_k \leqslant j < n_{k+1}} U_j$. We adapt this notion to one which is convenient for treating the Haver property, and selective screenability: We say a space has the *σ-disjoint grouping property* if for each sequence $(\mathcal{V}_n\colon n < \infty)$ such that:

(1) each $\mathcal{V}_n$ is a disjoint family of open sets and
(2) for each $x \in X$ there are infinitely many n with $x \in \bigcup \mathcal{V}_n$

there is a corresponding increasing sequence $n_1 < n_2 < \cdots < n_k < \cdots$ such that for each $x \in X$, for all but finitely many k, $x \in \bigcup(\bigcup_{n_k \leqslant i < n_{k+1}} \mathcal{V}_i)$.

Lemma 6. *The grouping property is equivalent to the σ-disjoint grouping property.*

Proof. To see that the grouping property implies the σ-disjoint grouping property, choose an infinite set of distinct points $x_1, x_2, \ldots, x_n, \ldots$ from X, and for each n put $U_n = (\bigcup \mathcal{V}_n) \setminus \{x_n\}$. Then each $x \in X$ is in infinitely many U_n, and so by the grouping property we can choose $n_1 < n_2 < \cdots < n_k < \cdots$ so that each $x \in X$ is in all but finitely many of the sets $\bigcup_{n_k \leqslant i < n_{k+1}} U_i$. But then the sequence of n_k's shows for the sequence of $\mathcal{V}_j$'s that X has the σ-disjoint grouping property. To see the converse implication, let $(U_n\colon n < \infty)$ be a sequence of open subsets of X which forms a large cover. By setting for each n $\mathcal{V}_n = \{U_n\}$, and applying the σ-disjoint grouping property to the sequence of $\mathcal{V}_n$'s, we see that X has the grouping property. □

We also use the following lemma often:

Lemma 7. *For a metric space (X, d) the following are equivalent*:

(1) *(X, d) has the Haver property.*
(2) *For each sequence $(\varepsilon_n\colon n < \infty)$ of positive real numbers there is a sequence $(\mathcal{U}_n\colon n < \infty)$ such that each $\mathcal{U}_n$ is a pairwise disjoint family of open sets, each of diameter less than ε_n, and $\bigcup_{n<\infty} \mathcal{U}_n$ is a large cover of X.*

Proof. We must prove that (1) $\Rightarrow$ (2). Let a sequence $\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_n > \cdots$ of positive real numbers be given and let $(I_j\colon j < \infty)$ be a partition of the set of positive integers into infinitely many infinite pairwise disjoint subsets I_j. We may assume that the sequence of ε_n's converges to 0.

Apply the fact that Y has the Haver property to each of the sequences $(\varepsilon_n\colon n \in I_j)$. For each m we find a sequence $(\mathcal{U}_n\colon n \in I_m)$ such that each $\mathcal{U}_n$ is a family of pairwise disjoint open sets, each of diameter less than ε_n, and such that $\bigcup_{n\in I_m} \mathcal{U}_n$ is a cover of Y. We claim that $\bigcup_{n<\infty} \mathcal{U}_n$ is a large cover of Y.

To see this, consider any $y \in Y$, and any n. Assume we have already selected sets $U_1, \dots, U_n$ such that for $j \leqslant n$ we have an m_j and $U_j \in \bigcup_{i\in I_{m_j}} \mathcal{U}_i$ with $y \in U_j$. We show how to find a $U_{n+1} \notin \{U_1, \dots, U_n\}$ with $y \in U_{n+1}$ and $U_{n+1} \in \bigcup_{k<\infty} \mathcal{U}_k$.

First, put $\delta = \min\{\mathrm{diam}(U_j)\colon j \leqslant n\}$ and $m = \max\{m_j\colon j \leqslant n\}$. Choose $k > m$ so large that $\varepsilon_k < \delta$. Then choose $\ell > m$ so large that $\min(I_\ell) > k$. Since $\bigcup_{i\in I_\ell} \mathcal{U}_i$ covers Y, choose U_{n+1} to be a member of this cover which contains y. By our choice of ℓ, $\mathrm{diam}(U_{n+1}) < \varepsilon_m < \delta$, and so $U_{n+1} \notin \{U_i\colon i \leqslant n\}$.

It follows that $\bigcup_{n<\infty} \mathcal{U}_n$ has infinitely many distinct members containing y. □

The following is an analogue of Theorem 1 of [15].

Theorem 8. *Let Y be a separable metric space with the Haver property and the σ-disjoint grouping property. Then for any metric space X which has the Haver property, $X \times Y$ has the Haver property.*

Proof. Let a sequence $\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_n > \cdots$ of positive real numbers be given. By Lemma 7 choose a sequence $(\mathcal{U}_n\colon n < \infty)$ where each $\mathcal{U}_n$ is a pairwise disjoint family of open sets, each of diameter less than ε_n, such that $\bigcup_{n<\infty} \mathcal{U}_n$ is a large cover of Y. By the σ-grouping property choose an increasing sequence $n_1 < n_2 < \cdots < n_k < \cdots$ such that for each $y \in Y$, for all but finitely many k, y is an element of $\bigcup(\bigcup_{n_k \leqslant j < n_{k+1}} \mathcal{U}_j)$.

For each k put $\delta_k = \varepsilon_{n_k}/2$. And then since X has the Haver property choose by Lemma 7 for each k a pairwise disjoint family $\mathcal{V}_k$ of open sets, each of diameter less than δ_k, such that $\bigcup_{k<\infty} \mathcal{V}_k$ is a large cover of X.

Now define for $i < n_1$: $\mathcal{W}_i = \{V \times U\colon V \in \mathcal{V}_1 \text{ and } U \in \mathcal{U}_i\}$. And for each k, and for $n_k \leqslant i < n_{k+1}$, define $\mathcal{W}_i = \{V \times U\colon V \in \mathcal{V}_{k+1} \text{ and } U \in \mathcal{U}_i\}$. Note that each $\mathcal{W}_i$ is a pairwise disjoint family of open subsets of $X \times Y$, and each member of $\mathcal{W}_i$ has diameter less than $\delta_i + \varepsilon_i/2$, which is less than ε_i, in the usual product metric. We have that $\bigcup_{i<\infty} \mathcal{W}_i$ is an open cover of $X \times Y$. To see that consider $(x, y) \in X \times Y$. Choose N so large that for each $k \geqslant N$ we have $y \in \bigcup(\bigcup_{n_k \leqslant i < n_{k+1}} \mathcal{U}_i)$. Then choose a $k > N$ with $x \in \bigcup \mathcal{V}_{k+1}$. It follows that (x, y) is in $\bigcup \mathcal{W}_i$ for some i with $n_k \leqslant i < n_{k+1}$. □

Corollary 9. *Let Y be a separable metric space with the Hurewicz property. If Y and X have the Haver property, then $X \times Y$ has the Haver property.*

Corollary 10. *Let Y be a separable metric space with the σ-disjoint grouping property. If Y has the Haver property, then all finite powers of Y have the Haver property.*

Corollary 11. *Let Y be a separable metric space which has the Haver property and the Hurewicz property. Then all finite powers of Y have the Haver property.*

Proof. By Lemma 3 of [15], the Hurewicz property implies the grouping property. By Lemma 6, the grouping property implies the σ-disjoint grouping property. Then by Corollary 10 all finite powers of Y have the Haver property. □

And these results imply directly:

Theorem 12. *If X is a metrizable space with the Haver property and if X^n has the Hurewicz property, then X^n has property $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$.*

Proof. By Corollary 11 all finite powers of X have the Haver property. Since X^n is Hurewicz, Theorem 1 implies that X^n has $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$. □

Corollary 13. *If X is a metric space and n a positive integer such that X satisfies $\mathsf{S}_c(\mathcal{O},\mathcal{O})$ and X^n satisfies the Hurewicz property, then X^n also satisfies $\mathsf{S}_c(\mathcal{O},\mathcal{O})$.*

Corollary 14. *If X and Y are metrizable spaces with $\mathsf{S}_c(\mathcal{O},\mathcal{O})$ and $Y \times X$ has the Hurewicz property, then $Y \times X$ has $\mathsf{S}_c(\mathcal{O},\mathcal{O})$.*

Proof. Since $X \times Y$ has the Hurewicz property, and this is preserved by closed subsets, X and Y individually have the Hurewicz property. By Corollary 11, $Y \times X$ has the Haver property. Since $Y \times X$ has the Hurewicz property, Theorem 1 implies $Y \times X$ has $\mathsf{S}_c(\mathcal{O},\mathcal{O})$. □

Next we obtain a product theorem where we remove the hypothesis that one of the factors have the Hurewicz property, but instead require countable dimensionality.

Theorem 15. *Let X and Y be metric spaces. If X has the Haver property and Y is countable dimensional then $X \times Y$ has the Haver property.*

Proof. Since a union of countably many subsets of a space, each with the Haver property, has the Haver property, it is enough to prove that if X has the Haver property and Y is zero-dimensional, then $X \times Y$ has the Haver property. Thus, let $(\varepsilon_n\colon n < \infty)$ be a sequence of positive real numbers. Since X has the Haver property choose for each n a pairwise disjoint family $\mathcal{V}_n$ of open subsets, each of diameter less than $\varepsilon_n/2$, such that $\bigcup_{n<\infty}\mathcal{V}_n$ covers X. Also, as Y is zero-dimensional, choose for each n a partition $\mathcal{W}_n$ of Y into disjoint open sets, each of diameter less than $\varepsilon_n/2$. Then, for each n define $\mathcal{U}_n$ to be the set $\{V \times W\colon V \in \mathcal{V}_n \text{ and } W \in \mathcal{W}_n\}$. Then each $\mathcal{U}_n$ is a pairwise disjoint family of open sets, each of diameter less than ε_n. And $\bigcup_{n<\infty}\mathcal{U}_n$ is a cover of $X \times Y$. □

By a theorem of Lelek, each complete metric space X has a metric compactification $B(X)$ such that $B(X) \setminus X$ is countable dimensional.

Theorem 16. *Let X be a complete metric space with the Haver property. For every metric space Y with the Haver property, $X \times Y$ has the Haver property.*

Proof. Let X be a complete metric space with the Haver property. Let $L(X)$ be Lelek's countable dimensional extension of X, resulting in the compact space $B(X)$.

Claim. *$B(X)$ has property $\mathsf{S}_c(\mathcal{O},\mathcal{O})$.*

For let a sequence $(\mathcal{U}_n\colon n < \infty)$ of open covers of $B(X)$ be given. For each n choose a pairwise disjoint open refinement $\mathcal{V}_{2\cdot n}$ of $\mathcal{U}_{2\cdot n}$ such that the countable dimensional space $L(X)$ is covered by $\mathcal{A} = \bigcup_{n<\infty}\mathcal{V}_{2\cdot n}$. Let $V \subset B(X)$ be the open set $\bigcup\mathcal{A}$. Then $B(X) \setminus V$ is closed, so compact, and is a subset of X, so has the Haver property. Since compactness implies the Hurewicz property, Theorem 5 implies that $B(X) \setminus V$ has property $\mathsf{S}_c(\mathcal{O},\mathcal{O})$. Applying this to the subsequence $(\mathcal{U}_{2\cdot n-1}\colon n < \infty)$ of the original sequence of open covers of $B(X)$, we find pairwise disjoint open refinements $\mathcal{V}_{2\cdot n-1}$ such that $\bigcup_{n<\infty}\mathcal{V}_{2\cdot n-1}$ is a cover of $B(X) \setminus V$. But then the sequence $(\mathcal{V}_n\colon n < \infty)$ of disjoint open refinements of $(\mathcal{U}_n\colon n < \infty)$ covers $B(X)$. This completes the proof of the claim.

Now $B(X)$ has both the Hurewicz and Haver properties, so by Theorem 8 $B(X) \times Y$ has the Haver property. But the Haver property is hereditary [6]. Then $X \times Y$ has the Haver property. □

In particular, we obtain that all examples constructed in [9] and [10] have the Haver property in all finite powers. Let $\mathbb{I}$ denote the space of irrational numbers endowed with the Baire metric. This is a complete, zero-dimensional, metric space. Thus the product of $\mathbb{I}$ with any metric space with the Haver property again has the Haver property. This shows that the products of the examples in [9] have the Haver property, though they fail to have property $\mathsf{S}_c(\mathcal{O},\mathcal{O})$. And the examples from Theorem 1 of [10] also have the Haver property in all finite powers. The reason is: The space X in that Theorem 1 is a complete metric space with the Haver property, and also $\mathbb{I}$ is a complete metric space with the Haver property. Thus all finite powers of $X \times \mathbb{I}$ have the Haver property, and so all subspaces of these finite powers have the Haver property.

This raises the following questions:

Problem 1. Is there a metric space X which has the Haver property, but $X \times X$ does not have the Haver property?

Problem 2. Is there a metric space X which has property $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$, but $X \times X$ does not have the Haver property?

3. New classes of weakly infinite dimensional spaces

In [3] the following classes of open covers are considered for infinite dimensional separable metric spaces X: Let CFD denote the collection of closed, finite dimensional subsets of X. Also, let FD denote the collection of finite dimensional subsets of X. Then $\mathcal{O}_{\mathsf{cfd}}$ denotes the open covers $\mathcal{U}$ of X such that $X \notin \mathcal{U}$, but for each $C \in \mathsf{CFD}$ there is a $U \in \mathcal{U}$ with $C \subseteq U$. And $\mathcal{O}_{\mathsf{fd}}$ denotes the open covers $\mathcal{U}$ of X such that $X \notin \mathcal{U}$, but for each $C \in \mathsf{FD}$ there is a $U \in \mathcal{U}$ with $C \subseteq U$. The games for the selection principles $\mathsf{S}_1(\mathcal{O}_{\mathsf{cfd}}, \mathcal{O})$ and $\mathsf{S}_1(\mathcal{O}_{\mathsf{fd}}, \mathcal{O})$ were considered there, and it was shown how these games characterize strong countable dimensional and countable dimensional spaces. From the monotonicity properties of selection principles it is also clear that

$$\mathsf{S}_1(\mathcal{O}_{\mathsf{cfd}}, \mathcal{O}) \Rightarrow \mathsf{S}_1(\mathcal{O}_{\mathsf{fd}}, \mathcal{O}).$$

We now show that also

$$\mathsf{S}_1(\mathcal{O}_{\mathsf{fd}}, \mathcal{O}) \Rightarrow \mathsf{S}_c(\mathcal{O}, \mathcal{O})$$

so that by a result of Addis and Gresham in [1], spaces satisfying $\mathsf{S}_1(\mathcal{O}_{\mathsf{cfd}}, \mathcal{O})$ and $\mathsf{S}_1(\mathcal{O}_{\mathsf{fd}}, \mathcal{O})$ are weakly infinite dimensional.

Theorem 17. *Let X be a metrizable space. If it has property $\mathsf{S}_1(\mathcal{O}_{\mathsf{fd}}, \mathcal{O})$, then it has property $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$.*

Proof. Let $(\mathcal{U}_n\colon n < \infty)$ be a sequence of open covers of X. Write $\mathbb{N} = \bigcup_{n\in\mathbb{N}} Y_n$ where each Y_n is infinite, and for $m \neq n$, $Y_m \cap Y_n = \emptyset$.

For each k, define $\mathcal{O}_k$ as follows: Let C be an n-dimensional subset of X for some finite n. By the Hurewicz–Tumarkin theorem (Theorem 3 in § 27.II of [8]), $C = C_1 \cup \cdots \cup C_{n+1}$ where each C_j is zero-dimensional. Consider any set $I \subset Y_k$ with $|I| = n+1$. Let $\{i_1, i_2, \ldots, i_{n+1}\}$ list I in increasing order. By [8], § 21.XI, Theorem 2, and § 26.II, Theorem 2, choose a pairwise disjoint refinement $\mathcal{V}_{i_j}$ of $\mathcal{U}_{i_j}$ which is a cover of C_j, and put $U(C, I) = \bigcup(\bigcup_{j\in I} \mathcal{V}_j)$. Put $\mathcal{O}_k = \{U(C, I)\colon C \subset X, I \subset Y_k \text{ finite and } \dim(C) + 1 = |I|\}$.

Each $\mathcal{O}_k$ is an element of $\mathcal{O}_{\mathsf{fd}}$. Apply $\mathsf{S}_1(\mathcal{O}_{\mathsf{fd}}, \mathcal{O})$ to the sequence $(\mathcal{O}_k\colon k < \infty)$. We find for each k a set $U_k \in \mathcal{O}_k$ such that $\{U_k\colon k < \infty\}$ is an open cover of X. For each k, choose a finite set $I_k \subset Y_k$ and a finite dimensional set $C_k \subset X$ with $|I_k| = \dim(C_k) + 1$, so that $U_k = U(C_k, I_k)$. For each $i \in I_k$, choose a pairwise disjoint refinement $\mathcal{V}_i$ of $\mathcal{U}_i$ such that $U_k = \bigcup(\bigcup_{i\in I_k} \mathcal{V}_i)$. For each $i \notin \bigcup_{k\in\mathbb{N}} I_k$, choose an arbitrary disjoint refinement $\mathcal{V}_i$ of $\mathcal{U}_j$.

The sequence $(\mathcal{V}_n\colon n < \infty)$ of pairwise disjoint refinements of $(\mathcal{U}_n\colon n < \infty)$ shows that $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$ holds.

The example given by R. Pol in [11] of a weakly infinite dimensional space which is not countable dimensional in fact has property $\mathsf{S}_1(\mathcal{O}_{\mathsf{cfd}}, \mathcal{O})$. To see this, recall that Pol's example X is of the form $X = M \cup L$ where M is a totally disconnected strongly infinite dimensional complete metric space, while L is countable dimensional, and X is compact. An examination of a proof of Lelek's compactification theorem by a countable dimensional extension shows we can take L to be strongly countable dimensional. Thus, take the version of Pol's example with L strongly countable dimensional. To see that X has $\mathsf{S}_1(\mathcal{O}_{\mathsf{cfd}}, \mathcal{O})$, let $(\mathcal{U}_n\colon n < \infty)$ be a sequence of elements of $\mathcal{O}_{\mathsf{cfd}}$. Write $L = \bigcup_{n\in\mathbb{N}} L_n$, where each L_n is closed and finite dimensional. For $n \in \mathbb{N}$ choose $U_{n+1} \in \mathcal{U}_{n+1}$ such that $L_n \subseteq U_{n+1}$. Then $X \setminus (\bigcup_{n\in\mathbb{N}} U_{n+1})$ is a closed, so compact, subset of the totally disconnected space M, and thus is zero-dimensional. Thus choose $U_1 \in \mathcal{U}_1$ to cover this zero-dimensional set.

To see that $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$ does not imply $\mathsf{S}_1(\mathcal{O}_{\mathsf{fd}}, \mathcal{O})$ we first prove a product preservation theorem:

Theorem 18. *Let X and Y be metrizable spaces.*

(1) *If X has property $\mathsf{S}_1(\mathcal{O}_{\mathsf{fd}}, \mathcal{O})$ and Y is countable dimensional, then $X \times Y$ has property $\mathsf{S}_1(\mathcal{O}_{\mathsf{fd}}, \mathcal{O})$.*
(2) *If X has property $\mathsf{S}_1(\mathcal{O}_{\mathsf{cfd}}, \mathcal{O})$ and Y is strongly countable dimensional, then $X \times Y$ has property $\mathsf{S}_1(\mathcal{O}_{\mathsf{cfd}}, \mathcal{O})$.*

Proof. We prove (1). The proof of (2) is similar. Write $Y = \bigcup_{n\in\mathbb{N}} Y_n$ where each Y_n is finite dimensional. Let $(\mathcal{U}_n\colon n \in \mathbb{N})$ be a sequence of FD-covers of $X \times Y$. Also write $\mathbb{N} = \bigcup_{k\in\mathbb{N}} S_k$ where each S_k is infinite and for $k \neq \ell$, S_k and S_ℓ are disjoint.

Fix k, and consider $(\mathcal{U}_n\colon n \in S_k)$. It is a sequence of FD-covers of $X \times Y$, and thus of $X \times Y_k$. By Menger's Theorem (see the Theorem in § 27 in Chapter II, Section VIII of [8]) for each finite dimensional set $C \subset X$, the set $C \times Y_k$ is finite dimensional and so for each $n \in S_k$ there is a $U \in \mathcal{U}_n$ with $C \times Y_k \subset U$. Put $\mathcal{V}_n := \{U \in \mathcal{U}_n\colon (\exists \text{ finite dimensional } C \subset X)(C \times Y_k \subseteq U)\}$. Then $(\mathcal{V}_n\colon n \in S_k)$ is a sequence of FD-covers for $X \times Y_k$. Applying $\mathsf{S}_1(\mathcal{O}_{\mathsf{fd}}, \mathcal{O})$ to the sequence $(\mathcal{V}_n\colon n \in S_k)$, choose for each $n \in S_k$ a set $U_n \in \mathcal{V}_n$ so that $\{U_n\colon n \in S_k\}$ covers $X \times Y_k$. Then $\{U_n\colon n \in \mathbb{N}\}$ is a cover of $X \times Y$. □

Corollary 19. $\mathsf{S}_c(\mathcal{O}, \mathcal{O}) \not\Rightarrow \mathsf{S}_1(\mathcal{O}_{\mathsf{fd}}, \mathcal{O})$.

Proof. In [10], E. Pol shows that here is a separable metric space X such that X has property $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$ and for some subspace Y of the space of irrational numbers (thus, a zero-dimensional space), $X \times Y$ is strongly infinite dimensional, and so not $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$. By Theorem 18, X is not a space with property $\mathsf{S}_1(\mathcal{O}_{\mathsf{fd}}, \mathcal{O})$. □

Since countable dimensional spaces have $\mathsf{S}_1(\mathcal{O}_{\mathsf{fd}}, \mathcal{O})$ but not necessarily $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O})$, it also follows that $\mathsf{S}_1(\mathcal{O}_{\mathsf{fd}}, \mathcal{O})$ does not imply $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O})$.

4. Open problems

In Fig. 1 the arrows denote implications, `SCD` denotes the class of strongly countable dimensional metric spaces and `CD` denotes the class of countable dimensional metric spaces.

The only new implication in this diagram is (3), proven in Theorem 17. We pointed out in Corollary 19 that (3) is not reversible. It is well known that the Hilbert Cube has the property $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O})$ but not $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$, and thus none of the implications to the top row is reversible. We pointed out that R. Pol's example in [11] of a weakly infinite dimensional space which is not countable dimensional can be taken in the class $\mathsf{S}_1(\mathcal{O}_{\mathsf{cfd}}, \mathcal{O})$, and thus none of the implications to the middle row is reversible. A classical example of Hurewicz also shows that the implication (1) is not reversible. And since the space of irrational numbers is in SCD but not in $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O})$, no implication to the middle column is reversible. The only two implications whose reversibility is not taken care of by these remarks are (2) and (4). We do not know the status of their reversibility. A positive answer to the following problem would show that these implications are not reversible:

Problem 3. Is there a countable dimensional separable metric space which does not have the selection property $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}_{\mathsf{cfd}}, \mathcal{O})$?

We have shown in Theorem 1 that if a metric space with the Haver property also has the Hurewicz property, then it has $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$. We also showed in Corollary 14 that if the product of two metrizable spaces, each with $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$, has the Hurewicz property, then this product has $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$. A prior example of R. Pol shows that here we cannot weaken the Hurewicz property to the Menger property $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O})$. But what about weakening the Hurewicz property

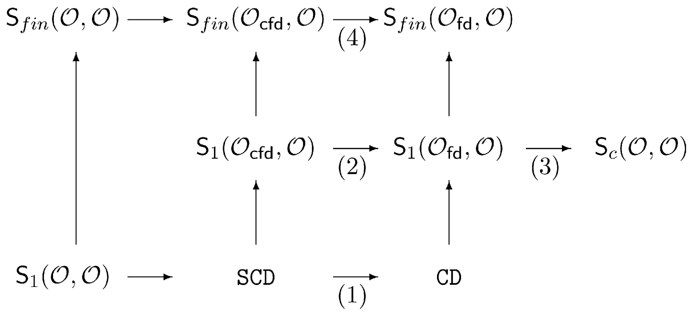

Fig. 1. Diagram of basic implications.

in Theorem 1 to the Menger property? Complete metric spaces with the Menger property have the Hurewicz property, and so:

Theorem 20. *If X is a complete metric space with the Menger property and the Haver property then it has* $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$.

Proof. Since X has the Menger property it is Lindelöf. Since X is metrizable and Lindelöf it is homeomorphic to a subspace of the Hilbert cube. Since X is completely metrizable it is by a classical theorem of Mazurkiewicz homeomorphic to a G_δ subset of the Hilbert cube. But continuous maps preserve the Menger property, and thus X is homeomorphic to a G_δ subset with the Menger property. By a classical theorem of Hurewicz (Theorem 20 of [7]), X is homeomorphic to a σ-compact subset of the Hilbert cube, and thus X is σ-compact. But then X has the Hurewicz property. Now apply Theorem 1. □

Problem 4. Can there be a metric space which has the Haver property and property $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O})$, but not the property $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$?

Finally, Theorem 5 prompts the following question:

Problem 5. If a metric space has the Haver property but not $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$, then is there an equivalent metric on this space in which the space does not have the Haver property?

Acknowledgement

The author wishes to thank the referee for a careful reading of the paper and constructive remarks. We originally had only (1) and (2) in the statement of Theorem 5: The referee pointed out that the proof really proves the implication (3) $\Rightarrow$ (1). Remarks by the referee also prompted the author to include Section 4.

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