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On some questions about selective separability

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CH implies that selective separability is not preserved by finite powers (solving [3, Problems 3.7 and 3.9]). In ZFC, selective separability does not imply H-separability (solving [4, Problem 34]).

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1 Definitions and notation

Let $\mathcal{A}$ and $\mathcal{B}$ be given families of subsets of some set S. Then the following symbols denote the corresponding statements for the pair $\mathcal{A}, \mathcal{B}$:

1. $\mathrm{S}_1(\mathcal{A},\mathcal{B})$: For each sequence $(O_m : m < \infty)$ of elements of $\mathcal{A}$ there is a sequence $(T_m : m < \infty)$ such that each T_m belongs to O_m, and $\{T_m : m < \infty\} \in \mathcal{B}$.
2. $\mathrm{S}_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$: For each sequence $(O_m : m < \infty)$ of elements of $\mathcal{A}$ there is a sequence $(T_m : m < \infty)$ such that each T_m is a finite subset of O_m, and $\bigcup\{T_m : m < \infty\} \in \mathcal{B}$.

For a given space X let $\mathcal{O}$ denote the collection of all its open covers and let Ω denote the collection of all its ω-covers (an open cover $\mathcal{U}$ of X is said to be an *ω-cover* if X is not a member of $\mathcal{U}$, but for each finite subset F of X there is $U \in \mathcal{U}$ such that $F \subset U$). Let $\mathcal{D}$ denote the collection of dense subsets of a T_3-space.

The symbol $\mathrm{C}_{\mathrm{p}}(X)$ denotes the space of all continuous real-valued functions on the space X in the topology of pointwise convergence.

We say that X *is selectively separable* [9] if it satisfies the selection principle $\mathrm{S}_{\mathrm{fin}}(\mathcal{D},\mathcal{D})$.

2 $\mathrm{S}_{\mathrm{fin}}(\Omega,\Omega)$ and direct sums

In [10] Scheepers constructed from CH subsets X and Y of ${}^{\omega}\mathbb{Z}$ each with the property $\mathrm{S}_1(\Omega,\Omega)$ such that

$$(X \cup Y) + (X \cup Y) = {}^{\omega}\mathbb{Z}.$$

It is well known that $\mathrm{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$ is preserved by continuous images, closed subsets, and countable unions, that $\mathrm{S}_{\mathrm{fin}}(\Omega,\Omega)$ is equivalent to $\mathrm{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$ in all finite powers, that $\mathrm{S}_{\mathrm{fin}}(\Omega,\Omega)$ is preserved by closed subsets, finite powers, and continuous images, and that ${}^{\omega}\mathbb{Z}$ (which is homeomorphic to the set of irrational numbers) does not have $\mathrm{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$. This implies that $(X \cup Y)^2$ does not have $\mathrm{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$ (the operation $+$ is continuous). Thus $X \cup Y$ does not have $\mathrm{S}_{\mathrm{fin}}(\Omega,\Omega)$ and also $X \times Y$ does not have $\mathrm{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$.

Theorem 2.1 (CH) *There are spaces X and Y each with the property $\mathrm{S}_1(\Omega,\Omega)$, but their topological sum $Z = X \oplus Y$ does not have $\mathrm{S}_{\mathrm{fin}}(\Omega,\Omega)$.*

Proof. Let X and Y be subsets of the set of reals each with the property $\mathrm{S}_1(\Omega,\Omega)$ such that

$$(X \cup Y) + (X \cup Y) = {}^{\omega}\mathbb{Z}.$$

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Let Z be the set $(X \times \{0\}) \cup (Y \times \{1\})$ with topology generated by

$$\{U \times \{0\} : U \subset_{\text{open}} X\} \cup \{V \times \{1\} : V \subset_{\text{open}} Y\}.$$

We write $Z = X \oplus Y$.

Then $Z \times Z$ is

$$\begin{aligned}\{((x_1,0),(x_2,0)) : x_1,x_2 \in X\} &\cup \{((x_1,0),(y_1,1)) : x_1 \in X, y_1 \in Y\}\\ &\cup \{((y_1,1),(x_1,0)) : y_1 \in Y, x_1 \in X\}\\ &\cup \{((y_1,1),(y_2,1)) : y_1,y_2 \in Y\}.\end{aligned}$$

Since $\{((x_1,0),(y_1,1)) : x_1 \in X, y_1 \in Y\}$, a closed subset of $Z \times Z$, is homeomorphic to $X \times Y$ it does not have $\mathrm{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$. Thus $Z \times Z$ does not have $\mathrm{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$. This implies that Z does not have $\mathrm{S}_{\mathrm{fin}}(\Omega,\Omega)$. □

Recall the following results from [1] and [9].

Theorem 2.2 [1] *If $Z = X \oplus Y$, then* $\mathrm{C_p}(Z) \cong \mathrm{C_p}(X) \times \mathrm{C_p}(Y)$.

Theorem 2.3 [9] *For X a separable metric space, the following are equivalent:*

1. X *has* $\mathrm{S}_{\mathrm{fin}}(\Omega,\Omega)$.
2. $\mathrm{C_p}(X)$ *has* $\mathrm{S}_{\mathrm{fin}}(\mathcal{D},\mathcal{D})$.

Theorem 2.4 [9] *For X a separable metric space, the following are equivalent:*

1. X *has* $\mathrm{S}_1(\Omega,\Omega)$.
2. $\mathrm{C_p}(X)$ *has* $\mathrm{S}_1(\mathcal{D},\mathcal{D})$.

Thus we have the following.

Corollary 2.5 (CH) *There are separable metric spaces X, Y such that* $\mathrm{C_p}(X), \mathrm{C_p}(Y)$ *both have* $\mathrm{S}_1(\mathcal{D},\mathcal{D})$, *but the product* $\mathrm{C_p}(X) \times \mathrm{C_p}(Y)$ *does not have* $\mathrm{S}_{\mathrm{fin}}(\mathcal{D},\mathcal{D})$.

Corollary 2.5 answers [3, Question 3.7 and Question 3.9].

3 $\mathrm{S}_{\mathrm{fin}}(\Omega,\Omega)$ and the Hurewicz property

A topological space X has *the Hurewicz property* [7] if for each sequence $(\mathcal{U}_n : n < \infty)$ of open covers of X there is a sequence $(\mathcal{V}_n : n < \infty)$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, and each element of X is in all but finitely many of the sets $\bigcup \mathcal{V}_n$.

A metric space (X, d) is *totally bounded* if there is for each $\varepsilon > 0$ a finite set $F \subset X$ such that

$$X \subseteq \bigcup_{f \in F} B_d(f, \varepsilon),$$

where $B_d(f,\varepsilon) = \{x \in X : d(x,f) < \varepsilon\}$. A metric space is *$\sigma$-totally bounded* if it is a union of countably many subsets, each totally bounded. In [4] the authors defined the following notion: The space X is H-*separable* if for every sequence of $(D_n : n < \infty)$ of elements of $\mathcal{D}$, there are finite sets $F_n \subset D_n$ such that for every open set U in X, the intersection $U \cap F_n$ is nonempty for all but finitely many n.

By [4, Theorem 40] we have that in a Tychonoff space X, $\mathrm{C_p}(X)$ is H-separable if and only if X^n has the Hurewicz property for each $n \in \mathbb{N}$.

Theorem 3.1 $\mathrm{S}_{\mathrm{fin}}(\mathcal{D},\mathcal{D})$ *does not imply* H-*separability.*

Before passing to a proof of the theorem, let us recall some results from [2] and [6].

Theorem 3.2 [2] *Let (X, d) be a metrizable space. The following are equivalent:*

1. *X has the Hurewicz property.*
2. *X is σ-totally bounded in each equivalent metric.*

Then we have the following lemma.

Lemma 3.3 *If X is a complete metric space and $Y \subset X$ has the Hurewicz property, then there is a σ-compact subset $Z \subset X$ with $Y \subset Z$.*

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Proof. Let X be a complete metric space and let $Y \subset X$ have the Hurewicz property. By Theorem 3.2, Y is σ-totally bounded with respect to the metric on X. So $Y = \bigcup_{n<\infty} Y_n$, where each Y_n is totally bounded. Since X is a complete metric space, the closure of each Y_n in X is compact (see [11, p. 182]). □

Theorem 3.4 [6] *In every Polish*[1] *space X without any compact set with nonempty interior there is $M \subset X$ whose finite powers all have $\mathrm{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O})$, but M is not contained in any σ-compact subset of X.*

Proof of Theorem 3.1. The space ${}^{\omega}\mathbb{Z}$ is a complete separable metric space and no compact subset contains a nonempty open subset of ${}^{\omega}\mathbb{Z}$. By Theorem 3.4 there is a subset M of ${}^{\omega}\mathbb{Z}$ with the property $\mathrm{S}_{\mathrm{fin}}(\Omega, \Omega)$ that is not contained in any σ-compact subset of ${}^{\omega}\mathbb{Z}$. By Lemma 3.3, M does not have the Hurewicz property. By [4, Theorem 40], $\mathrm{C}_{\mathrm{p}}(M)$ is not H-separable. By Theorem 2.3, $\mathrm{C}_{\mathrm{p}}(M)$ has property $\mathrm{S}_{\mathrm{fin}}(\mathcal{D}, \mathcal{D})$. □

Theorem 3.1 answers [4, Question 34].

4 Remarks

The statement of Corollary 2.5 is independent of ZFC. The reasons are: By Theorem 2.4 these X and Y must have the property $\mathrm{S}_1(\Omega, \Omega)$. But $\mathrm{S}_1(\Omega, \Omega)$ implies Borel's property of strong measure zero. By a result of R. Laver [8], Borel's Conjecture that such sets of reals are countable, is consistent. By a result of T. Carlson [5], Borel's Conjecture implies that all separable metric spaces with strong measure zero are countable. But if X and Y are countable, so is $Z = X \oplus Y$, and so $\mathrm{C}_{\mathrm{p}}(X) \times \mathrm{C}_{\mathrm{p}}(Y)$ still has property $\mathrm{S}_1(\mathcal{D}, \mathcal{D})$.

I suspect that the answer to the following problem is "yes".

Problem 4.1 *Are there in ZFC separable metric spaces X and Y such that $\mathrm{C}_{\mathrm{p}}(X)$ and $\mathrm{C}_{\mathrm{p}}(Y)$ both have property $\mathrm{S}_{\mathrm{fin}}(\mathcal{D}, \mathcal{D})$, but $\mathrm{C}_{\mathrm{p}}(X) \times \mathrm{C}_{\mathrm{p}}(Y)$ does not have property $\mathrm{S}_{\mathrm{fin}}(\mathcal{D}, \mathcal{D})$?*

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■ Please regard page 260, items 1. and 2. We changed the text in both items. In 2., I changed $\cup$ to $\bigcup$ (similarly in line 3 of Section 3 on page 261 and in line 2 of page 262). Moreover, I wonder whether $\{T_m : m < \infty\} \in \mathcal{B}$ in 1. should also read $\bigcup\{T_m : m < \infty\} \in \mathcal{B}$ as in 2.

■ Please regard page 260, the first displayed formula. It should be explained what $(X \cup Y) + (X \cup Y)$ stands for, possibly in Section 1. The same holds for $(X \cup Y)^2$, see two lines before Theorem 2.1.

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[1] i. e., completely metrizable, separable

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