Very good! 9/9

[redacted]

Extra Credit #3
October 30, 2009

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21.4 Prove the following by induction. In each case, n is a positive integer

a) $2^n \le 2^{n+1} - 2^{n-1} - 1$

Proof:

n=1: $\underbrace{2^1 = 2}_{\text{left side}}$ $\quad \underbrace{2^{1+1} - 2^{1-1} - 1 = 2^2 - 2^0 - 1 = 2^2 - 2 = 4 - 2 = 2}_{\text{right side}}$

The inequality holds for n=1 since $2 \le 2$.

n=k: We will assume that the inequality holds for n=k. So $2^k \le 2^{k+1} - 2^{k-1} - 1$

n=k+1: $2^{k+1} = 2 \cdot 2^k \le 2(2^{k+1} - 2^{k-1} - 1)$.

$$2(2^{k+1} - 2^{k-1} - 1) = 2^{k+2} - 2^{k-1+1} - 2 =$$
$$(2^{k+2} - 2^k - 1) - 1 \le 2^{k+2} - 2^k - 1, \text{ since}$$

$2^{k+2} - 2^k - 1 \ge 2^{k+2} - 2^k - 2$ we can say that $2^{k+1} \le 2^{k+2} - 2^k - 1$

We proved that $2^{k+1} \le 2^{k+2} - 2^k - 1$

Therefore, by induction we proved that the inequality is true for all positive integers n. ■

B) $(1-\frac{1}{2})(1-\frac{1}{4})(1-\frac{1}{8}) \cdots (1-\frac{1}{2^n}) \ge \frac{1}{4} + \frac{1}{2^{n+1}}$

Proof:

n=1: $\underbrace{(1-\frac{1}{2}) = \frac{1}{2}}_{\text{left}}$ $\quad \underbrace{\frac{1}{4} + \frac{1}{2^{1+1}} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}}_{\text{right}}$

$\frac{1}{2} \ge \frac{1}{2}$ so the inequality holds for n=1.

n=k: We will assume that the inequality holds for n=k. So $(1-\frac{1}{2})(1-\frac{1}{4}) \cdots (1-\frac{1}{2^k}) \ge \frac{1}{4} + \frac{1}{2^{k+1}}$

Very good!

$n=k+1: \underbrace{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\dots\left(1-\frac{1}{2^k}\right)}_{\text{this is case } n=k}\left(1-\frac{1}{2^{k+1}}\right)$

so we have that

$\geq \left(\frac{1}{4}+\frac{1}{2^{k+1}}\right)\left(1-\frac{1}{2^{k+1}}\right) \geq \frac{1}{4}-\frac{1}{4(2^{k+1})}+\frac{1}{2^{k+1}}-\frac{1}{2^{k+1}(2^{k+1})} =$

$= \frac{1}{4}-\frac{1}{4(2^{k+1})}+\frac{4}{4(2^{k+1})}-\frac{1}{2^{2k+2}} = \frac{1}{4}+\frac{3}{4(2^{k+1})}-\frac{1}{2^{2k+2}} =$

$= \frac{1}{4}+\frac{3}{2^{k+3}}-\frac{1}{2^{2k+2}} = \frac{1}{4}+\frac{1}{2^{k+2}}\left(\frac{3}{2}-\frac{1}{2^{k+1}}\right) \dots (*)$

(**) ... Note $\left(\frac{3}{2}-\frac{1}{2^{k+1}}\right) \geq 1$ for all positive values of k

Since $\frac{-1}{2^{k+1}} \geq -\frac{1}{2}$ $\quad \frac{1}{2^{k+1}} \leq \frac{1}{2}$ $\quad 2^{k+1} \geq 2^k$ which we know is true since $k>0$.

We therefore have that since (*) showed $(1-1/2)(1-1/4)\dots(1-\frac{1}{2^{k+1}}) \geq \frac{1}{4}+\frac{1}{2^{k+2}}\left(3/2-\frac{1}{2^{k+1}}\right)$ and therefore by (**)

$\frac{1}{4}+\frac{1}{2^{k+2}}\left(\frac{3}{2}-\frac{1}{2^{k+1}}\right) \geq \frac{1}{4}+\frac{1}{2^{k+2}}$. Therefore the statement is true for $n=k+1$. By induction principle we have that the statement is true for all $n>0$. ∎

C) $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots+\frac{1}{2n} \geq 1+\frac{n}{2}$

$n=1$: $\underbrace{1+\frac{1}{2}=3/2}_{\text{right side}} \geq \underbrace{1+1/2=3/2}_{\text{left side}}$

Therefore the statement is true for $n=1$.

$n=k$: Assume that the statement is true for $n=k$.

ie: $1+\frac{1}{2}+1/3+1/4+\dots\frac{1}{2^k} \geq 1+\frac{k}{2}$

$n=k+1$: $\underbrace{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots\frac{1}{2^k}}_{\text{case when } n=k}+\frac{1}{2^k+1}+\frac{1}{2^k+2}\dots\frac{1}{2^{k+1}} \geq$

$$\geq 1+\frac{k}{2}+\frac{1}{2^k+1}+\frac{1}{2^k+2}+\dots\frac{1}{2^{k+1}} \quad \dots (A)$$

Let $(*)=\frac{1}{2^k+1}+\frac{1}{2^k+2}+\dots\frac{1}{2^{(k+1)}}$ ok!

The number of terms in $(*)$ is $2^{k+1}-2^k$ — $4-2=2$

$2^k(2^1-1)=2^k$ terms

$\frac{1}{2^k+1}\geq\frac{1}{2^{(k+1)}}$ since $2^{k+1}\geq 2^k+1$

since $k\geq 1$. This tells us that each term in $(*)$ is greater than or equal to $\frac{1}{2^{(k+1)}}$ since $\frac{1}{2^k+1}$ is the smallest term in $(*)$.

Therefore:

$$(*)\geq\underbrace{\frac{1}{2^{k+1}}+\frac{1}{2^{k+1}}+\frac{1}{2^{k+1}}\dots}_{2^k \text{ times}}\geq\frac{2^k}{2^{k+1}}\geq\frac{1}{2^1} \quad \dots (B)$$

By (B) we have that

$$1+\frac{k}{2}+\frac{1}{2^k+1}+\dots\frac{1}{2^{(k+1)}}\geq 1+\frac{k}{2}+\frac{1}{2} \dots (C)$$

By (C) and (A) we have that

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots\frac{1}{2^k}\geq 1+\frac{k}{2}+\frac{1}{2}\geq 1+\frac{k+1}{2}$$

Therefore the statement is true for $n=k+1$.

By induction principle we know that the statement is true for all $n>0$. $\blacksquare$

$n=k$. So $(1-\frac{1}{2})(1-\frac{1}{4})\dots(1-\frac{1}{2^k})\geq\frac{1}{4}+\frac{1}{2^{k+1}}$