SELECTIVE SCREENABILITY AND THE HUREWICZ PROPERTY

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Abstract. We characterize the Hurewicz covering property in metrizable spaces in terms of properties of the metrics of the space - Theorem 1. Then we show that a weak version of selective screenability, when combined with the Hurewicz property, implies selective screenability - Theorem 4.

1. Definitions and notation

Let X be an infinite set, and let $\mathcal{A}$ and $\mathcal{B}$ be collections of families of subsets of X. The selection principle $\mathsf{S}_c(\mathcal{A},\mathcal{B})$, introduced in [2], states:

> For each sequence $(A_n : n < \infty)$ of elements of the family $\mathcal{A}$ there exists a sequence $(B_n : n < \infty)$ such that for each n B_n is a pairwise disjoint family refining A_n, and $\bigcup_{n<\infty} B_n$ is a member of the family $\mathcal{B}$.

For X topological space $\mathcal{O}$ denotes the collection of all open covers of X and $\mathcal{O}_{fin}$ denotes the collection of all finite open covers of X. For a positive integer n let $\mathcal{O}_n$ denote the collection of open covers consisting of at most n sets. Addis and Gresham introduced the instance $\mathsf{S}_c(\mathcal{O},\mathcal{O})$ of the selection principle in [1], where it was called property C. It is a selective version of the screenability property introduced by Bing in [4].

As was shown in [1], $\mathsf{S}_c(\mathcal{O},\mathcal{O})$ is a natural generalization of finite covering dimension to the infinite. Alexandroff's notion of weakly infinite dimensional is also a natural generalization of finite covering dimension, and is equivalent to $\mathsf{S}_c(\mathcal{O}_2,\mathcal{O})$. Hurewicz's notion of countable dimensionality is another natural generalization of finite covering dimension: X is countable dimensional if it is a union of countably may finite dimensional subspaces. The following implications hold - see [1]:

$$\text{countable dimensional} \Rightarrow \mathsf{S}_c(\mathcal{O},\mathcal{O}) \Rightarrow \mathsf{S}_c(\mathcal{O}_{fin},\mathcal{O}) \Rightarrow \mathsf{S}_c(\mathcal{O}_2,\mathcal{O}).$$

The Hilbert cube, $[0,1]^{\mathbb{N}}$, does not have property $\mathsf{S}_c(\mathcal{O}_2,\mathcal{O})$ - [1]. Borst proved in [6] that there exists a compact separable metric space X which has property $\mathsf{S}_c(\mathcal{O}_2,\mathcal{O})$, but not property $\mathsf{S}_c(\mathcal{O},\mathcal{O})$. Since for compact spaces

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$\mathsf{S}_c(\mathcal{O}_{fin},\mathcal{O}) \Leftrightarrow \mathsf{S}_c(\mathcal{O},\mathcal{O})$, Borst's example shows that $\mathsf{S}_c(\mathcal{O}_2,\mathcal{O})$ does not imply $\mathsf{S}_c(\mathcal{O}_{fin},\mathcal{O})$. R. Pol gave in [14] a compact metric space which has property $\mathsf{S}_c(\mathcal{O},\mathcal{O})$ but is not countable dimensional. It is an open problem if $\mathsf{S}_c(\mathcal{O}_{fin},\mathcal{O})$ implies $\mathsf{S}_c(\mathcal{O},\mathcal{O})$ - see Question 3.10 of [5]. We expect that the answer to this question is "No", and state a conjecture about it near the end of this paper. In [5] a class of spaces which do not distinguish $\mathsf{S}_c(\mathcal{O}_{fin},\mathcal{O})$ and $\mathsf{S}_c(\mathcal{O},\mathcal{O})$ is identified. In this paper we will extend this to a larger class of separable metric spaces which do not distinguish $\mathsf{S}_c(\mathcal{O}_{fin},\mathcal{O})$ and $\mathsf{S}_c(\mathcal{O},\mathcal{O})$. Examples show that the class we describe properly extends the class from [5].

In Section 2 we first give a convenient characterization of the Hurewicz property in metrizable spaces. In Section 3 we show that metrizable spaces with the Hurewicz property do not distinguish $\mathsf{S}_c(\mathcal{O}_{fin},\mathcal{O})$ and $\mathsf{S}_c(\mathcal{O},\mathcal{O})$. In Section 4 we connect this with Borst's work from [5] and in the final section we state a conjecture.

2. Characterizing the Hurewicz property in metrizable spaces.

A topological space X has the *Hurewicz property* [11] if there is for each sequence $(\mathcal{U}_n : n < \infty)$ of open covers of X a sequence $(\mathcal{V}_n : n < \infty)$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, and each element of X is in all but finitely many of the sets $\cup\mathcal{V}_n$. The metrizable space X is said to be *Haver* [9] with respect to a metric d if there is for each sequence $(\epsilon_n : n < \infty)$ of positive reals a sequence $(\mathcal{V}_n : n < \infty)$ where each $\mathcal{V}_n$ is a pairwise disjoint family of open sets, each of d-diameter less than ϵ_n, such that $\bigcup_{n<\infty}\mathcal{V}_n$ is a cover of X.

A metric space (X,d) is totally bounded if there is for each $\epsilon > 0$ a finite set $F \subset X$ such that $X \subseteq \bigcup_{f\in F} B_d(f,\epsilon)$, where $B_d(f,\epsilon) = \{x \in X : d(x,f) < \epsilon\}$. A metric space is σ-totally bounded if it is a union of countably many subsets, each totally bounded.

Theorem 1. *Let (X,d) be a metrizable space. The following are equivalent:*

1. *X has the Hurewicz property.*
2. *X is σ-totally bounded in each equivalent metric.*

Proof: $1 \Rightarrow 2$: For each n let $\delta_n = (1/2)^n$ and $\mathcal{U}_n = \{B_d(x,\delta_n) : x \in X\}$ where d is an arbitrary fixed metric of X. Apply the Hurewicz property to $(\mathcal{U}_n : n < \infty)$. For each n choose a finite set $\mathcal{V}_n \subset \mathcal{U}_n$ such that each $x \in X$ is in all but finitely many of the sets $\cup\mathcal{V}_n$. For each n define $X_n = \bigcap_{m\geq n}\cup\mathcal{V}_m$. Then for each n, and for $m \leq n$, $X_m \subseteq X_n$ and $\bigcup_{n<\infty} X_n$ covers X. We show that each X_n is totally bounded in the metric d: Consider an $\epsilon > 0$, and consider any X_n. Choose $m > n$ so large that $2\cdot(1/2)^m \leq \epsilon$. Each element of $\mathcal{V}_m$ is an open set of diameter less than $2\cdot(1/2)^m$, and $\mathcal{V}_m$ is a finite cover of X_n.
$2 \Rightarrow 1$: Let $(\mathcal{U}_n : n < \infty)$ be a sequence of open covers of X. By Remark 4,

page 196 from [8] let d be a metric generating the topology of X such that for each n, $\mathcal{W}_n = \{B_d(x, 1/n) : x \in X\}$ refines $\mathcal{U}_n$. Write $X = \bigcup_{n<\infty} X_n$, where each X_n is totally bounded and $X_n \subset X_{n+1}$ for every n. Choose for each m a finite $\mathcal{F}_m \subset \mathcal{W}_m$ with $X_m \subseteq \cup\mathcal{F}_m$. Then, for each m choose a finite $\mathcal{V}_m \subset \mathcal{U}_m$ such that $\mathcal{F}_m$ refines $\mathcal{V}_m$. Then, for each $x \in X$ for all but finitely many n, $x \in \cup\mathcal{V}_n$. $\diamondsuit$

3. $\mathsf{S}_c(\mathcal{O}_{fin}, \mathcal{O})$ in metrizable spaces with the Hurewicz property.

For easy reference, we denote the following strong form of $\mathsf{S}_c(\mathcal{O}_{fin}, \mathcal{O})$ by the symbol $\mathsf{S}_c^+(\mathcal{O}_{fin}, \mathcal{O})$:

> For each sequence $(\mathcal{U}_n : n < \infty)$ of finite open covers of X there are a sequence $(\mathcal{W}_n : n < \infty)$ and a sequence $m_1 < m_2 < ... < m_k < ...$ such that
> (1) each $\mathcal{W}_n$ is a finite pairwise disjoint family of open sets,
> (2) each $\mathcal{W}_n$ refines $\mathcal{U}_n$ and
> (3) for each $x \in X$, for all but finitely many k there is a $j \in [m_k, m_{k+1})$ with $x \in \cup\mathcal{W}_j$.

Lemma 2. *Let (X, d) be a metrizable space. If X has $\mathsf{S}_c(\mathcal{O}_{fin}, \mathcal{O})$ and the Hurewicz property then it has the property $\mathsf{S}_c^+(\mathcal{O}_{fin}, \mathcal{O})$.*

Proof: Recall that X has the Hurewicz property if and only if ONE has no winning strategy in the Hurewicz game (Theorem 27 of [15]). Let $(\mathcal{U}_n : n < \infty)$ be a sequence of finite open covers of X. Applying $\mathsf{S}_c(\mathcal{O}_{fin}, \mathcal{O})$ to $(\mathcal{U}_n : n < \infty)$, choose for each n a pairwise disjoint refinement $\mathcal{V}_n^1$ of $\mathcal{U}_n$ so that $F(\emptyset) = \bigcup_{n<\infty} \mathcal{V}_n^1$ covers X. This defines ONE's first move in the Hurewicz game. When TWO chooses a finite $T_1 \subset F(\emptyset)$, define $m_1 = min\{n : T_1 \subseteq \bigcup_{j<n} \mathcal{V}_j^1\}$. Next, apply $\mathsf{S}_c(\mathcal{O}_{fin}, \mathcal{O})$ to $(\mathcal{U}_n : n \geq m_1)$ and choose for each $n \geq m_1$ a pairwise disjoint $\mathcal{V}_n^2$ that refines $\mathcal{U}_n$ consisting of open sets, so that $F(T_1) = \bigcup_{n \geq m_1} \mathcal{V}_n^2$ covers X. This defines ONE's response to TWO's move T_1. When TWO chooses a finite $T_2 \subset F(T_1)$, define $m_2 = min\{n : T_2 \subseteq \bigcup_{m_1 \leq j < n} \mathcal{V}_j^2\}$ and apply $\mathsf{S}_c(\mathcal{O}_{fin}, \mathcal{O})$ to $(\mathcal{U}_n : n \geq m_2)$ to define $F(T_1, T_2)$, and so on.

Since X has the Hurewicz property F is not a winning strategy for ONE. Consider an F-play $F(\emptyset), T_1, F(T_1), T_2, F(T_1, T_2), T_3...$ lost by ONE. Then each T_m is finite and each $x \in \cup T_m$ for all but finitely many m. For $j < m_1$ define $\mathcal{W}_j = \{T \in T_1 : T \in \mathcal{V}_j^1\}$. For $m_k \leq j < m_{k+1}$ define $\mathcal{W}_j = \{T \in T_{k+1} : T \in \mathcal{V}_j^{k+1}\}$. For each j, $\mathcal{W}_j$ is finite pairwise disjoint and refines $\mathcal{U}_j$. $\diamondsuit$

Theorem 3. *If (X, d) is σ-totally bounded and has property $\mathsf{S}_c^+(\mathcal{O}_{fin}, \mathcal{O})$, then X has the Haver property in d.*

Proof: Write $X = \bigcup_{n<\infty} X_n$, where each $X_n \subset X$ is d-totally bounded and $X_n \subset X_{n+1}$. Let $(\epsilon_n : n < \infty)$ be a sequence of positive reals. By replacing ϵ_n's if necessary, we may assume that always $\epsilon_{n+1} < \frac{1}{2} \cdot \epsilon_n$. For each n, put $\delta_n = \frac{2^n-1}{2^n} \cdot (\frac{1}{3} \cdot \epsilon_n)$. For each n, choose a finite set $F_n \subset X_n$ such that $\{B(x,\delta_n) : x \in F_n\}$ covers X_n, and put $\mathcal{U}_n = \{B(x, \frac{1}{3} \cdot \epsilon_n) : x \in F_n\} \bigcup \{X \setminus \bigcup \{\overline{B(x,\delta_n)} : x \in F_n\}$, a finite open cover of X. Observe that for each n, $\overline{B(x,\delta_n)} \subset B(x,\epsilon_n)$, and $X_n \bigcap (X \setminus \bigcup \{\overline{B(x,\delta_n)} : x \in F_n\}) = \emptyset$.

Apply $\mathsf{S}_c^+(\mathcal{O}_{fin}, \mathcal{O})$ to the sequence $(\mathcal{U}_n : n < \infty)$. For each n find a finite pairwise disjoint refinement $\mathcal{H}'_n$ of $\mathcal{U}_n$ and find a sequence $m_1 < m_2 < ... < m_k < ...$ such that for each $x \in X$ for all but finitely many k, there is a j with $m_k \le j < m_{k+1}$ and $x \in \cup \mathcal{H}'_j$. Now for each n, put

$$\mathcal{H}_n = \{V \in \mathcal{H}'_n : (\exists x \in F_n)(V \subseteq B(x, \frac{1}{2} \cdot \epsilon_n))\}.$$

Claim: $\bigcup_{n<\infty} \mathcal{H}_n$ covers X.
For consider $x \in X$. Choose N so large so that for all $n \ge N$, $x \in X_n$ and for all $m_k \ge N$, there is $j \in [m_k, m_{k+1})$ with $x \in \cup \mathcal{H}'_j$. Choose k with $m_k \ge N$ and j with $m_k \le j < m_{k+1}$ with $x \in V$ for some $V \in \mathcal{H}'_j$. We have that $x \in X_j$, so V is not a subset of $X \setminus (\bigcup \{\overline{B(y,\delta_j)} : y \in F_j\})$ which means that $V \in \mathcal{H}_j$.

Since the diameter of any element of an $\mathcal{H}_n$ is less than ϵ_n, the sequence $(\mathcal{H}_n : n < \infty)$ witnesses the Haver property of X for $(\epsilon_n : n < \infty)$. $\diamondsuit$

Note that the Hurewicz property plus $\mathsf{S}_c(\mathcal{O}_2, \mathcal{O})$ does not imply the Haver property: For if this were to imply the Haver property, then by Theorem 1 of [3] it would follow that $\mathsf{S}_c(\mathcal{O}_2, \mathcal{O})$ plus the Hurewicz property implies $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$. Compactness implies the Hurewicz property, and [6] shows that $\mathsf{S}_c(\mathcal{O}_2, \mathcal{O})$ plus compact does not imply $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$.

Theorem 4. *If X is a metrizable space and has the Hurewicz property, then the following are equivalent:*

(1) *X has $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$*
(2) *X has $\mathsf{S}_c(\mathcal{O}_{fin}, \mathcal{O})$*

Proof: $1 \Rightarrow 2$: It is clear.
$2 \Rightarrow 1$: By the previous theorem X has the Haver property. By Theorem 1 from [3] we have that X has $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$. $\diamondsuit$

4. An extension of the class of "finite C-spaces".

In §3 of [5], Borst introduces the notion of a "finite C-space": A topological space X is a *finite C-space* if there is for each sequence $(\mathcal{U}_n : n < \infty)$ of finite open covers of X an n, and a sequence $(\mathcal{V}_j : j \le n)$ such that each $\mathcal{V}_j$ is a disjoint refinement of $\mathcal{U}_j$, and $\bigcup_{j \le n} \mathcal{V}_j$ is an open cover of X. And a space X is said to have "property K" if it has a compact subset C such that for every open subset U of X with $C \subset U$, the set $X \setminus U$ is finite dimensional. And in Theorem 3.8 of [5] the following equivalence is proved:

Theorem 5 (Borst)**.** *For separable metric spaces X the following are equivalent:*

(1) *X is a finite C-space.*

(2) *X has $\mathsf{S}_c(\mathcal{O},\mathcal{O})$ and property K.*

Thus, also in the class of spaces with property K, $\mathsf{S}_c(\mathcal{O},\mathcal{O})$ is equivalent to $\mathsf{S}_c(\mathcal{O}_{fin},\mathcal{O})$. And there are spaces with property K and $\mathsf{S}_c(\mathcal{O},\mathcal{O})$ which do not have the Hurewicz property: Let C be the the compact metric space from [14]: It has property $\mathsf{S}_c(\mathcal{O},\mathcal{O})$ and is infinite dimensional. Let P be the space of irrational numbers. Then X, the topological sum of C and P, has $\mathsf{S}_c(\mathcal{O},\mathcal{O})$ and property K. It is well known that the closed subset P of X does not have the Hurewicz property, and so X does not have the Hurewicz property.

As pointed out in [5], the space K_ω consisting of the elements x of $[0,1]^{\mathbb{N}}$ for which $x(n) > 0$ for only finitely many n is not a "finite C-space": For if it were a finite C-space, then by Theorem 1.2 of [5] it has a compactification with property $\mathsf{S}_c(\mathcal{O},\mathcal{O})$. But no compactification of K_ω has the property $\mathsf{S}_c(\mathcal{O},\mathcal{O})$: See example 5.3.6 in [7]. But K_ω is σ-compact and so has the Hurewicz property, and it is countable dimensional, so has property $\mathsf{S}_c(\mathcal{O},\mathcal{O})$.

Corollary 6. *Let X be a separable metric space which has an F_σ subset C such that: C has the Hurewicz property, and for every open set $U \subset X$ with $C \subset U$, $X \setminus U$ has $\mathsf{S}_c(\mathcal{O},\mathcal{O})$. Then the following are equivalent:*

(1) *X has the property $\mathsf{S}_c(\mathcal{O},\mathcal{O})$.*
(2) *X has the property $\mathsf{S}_c(\mathcal{O}_{fin},\mathcal{O})$.*

The proof uses the fact that $\mathsf{S}_c(\mathcal{O}_{fin},\mathcal{O})$ and $\mathsf{S}_c(\mathcal{O},\mathcal{O})$ are preserved by F_σ-subsets.

5. Remarks

In [10] Hurewicz introduced a property weaker than the Hurewicz property, and known as Menger's property: For each sequence $(\mathcal{U}_n : n < \infty)$ of open covers of a space X there is a sequence $(\mathcal{V}_n : n < \infty)$ of finite sets such that for each n, $\mathcal{V}_n \subset \mathcal{U}_n$, and $\bigcup_{n<\infty} \mathcal{V}_n$ is a cover of X. Theorem 3 shows that if a metrizable space has the Hurewicz property and also $\mathsf{S}_c(\mathcal{O}_{fin},\mathcal{O})$, then it has the Haver property. We have the following conjecture:

Conjecture 1. *There is a metrizable space X with the Menger property and $\mathsf{S}_c(\mathcal{O}_{fin},\mathcal{O})$, which does not have the Haver property in some metric.*

Note that Conjecture 1 implies that the answer to Borst's Question 3.10 is "no".

We also expect that for each $n > 1$ the implication $\mathsf{S}_c(\mathcal{O}_n,\mathcal{O}) \Rightarrow \mathsf{S}_c(\mathcal{O}_{n+1},\mathcal{O})$ is false.

In Remark D of [13] E. and R. Pol showed that a metrizable space has the property $\mathsf{S}_c(\mathcal{O},\mathcal{O})$ if, and only if, it has the Haver property in all equivalent metrics. This gives another way to conclude Theorem 4: By Theorems 1 and 3, we see that the Hurewicz property and $\mathsf{S}_c(\mathcal{O}_{fin},\mathcal{O})$ implies the Haver property for all equivalent metrics. Also: By [12] Remark D, Conjecture

1 translates to statement that Theorem 4 fails if the Hurewicz property is replaced with the Menger property.

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