SELECTIVE SCREENABILITY GAME
AND COVERING DIMENSION

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ABSTRACT. We introduce an infinite two-person game inspired by the selective version of R. H. Bing’s notion of screenability. We show how, for metrizable spaces, this game is related to covering dimension.

1. Introduction

Let $X$ be a topological space. In [3], R. H. Bing introduced the following notion of screenability: For each open cover $\mathcal{U}$ of $X$ there is a sequence $(\mathcal{V}_n : n < \infty)$ such that for each $n$, $\mathcal{V}_n$ is a family of pairwise disjoint open sets; for each $n$, $\mathcal{V}_n$ refines $\mathcal{U}$ and $\bigcup_{n<\infty} \mathcal{V}_n$ is an open cover of $X$. In [1], David F. Addis and John H. Gresham introduced the selective version of screenability: For each sequence $(\mathcal{U}_n : n < \infty)$ of open covers of $X$ there is a sequence $(\mathcal{V}_n : n < \infty)$ such that for each $n$, $\mathcal{V}_n$ is a family of pairwise disjoint open sets; for each $n$, $\mathcal{V}_n$ refines $\mathcal{U}_n$ and $\bigcup_{n<\infty} \mathcal{V}_n$ is an open cover of $X$. It is evident that selective screenability implies screenability.

Selective screenability is an example of the following selection principle which was introduced in [2]: Let $S$ be a set and let $\mathcal{A}$ and $\mathcal{B}$ be families of collections of subsets of the set $S$. Then $S_c(\mathcal{A}, \mathcal{B})$

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Thus, if $\mathcal{U}$ is a member of $\mathcal{A}$ or of $\mathcal{B}$, then $\mathcal{U}$ is a collection of subsets of $S$. 1
denotes the statement that for each sequence \((\mathcal{U}_n : n < \infty)\) of elements of \(\mathcal{A}\) there is a sequence \((\mathcal{V}_n : n < \infty)\) such that

1. for each \(n\), \(\mathcal{V}_n\) is a family of pairwise disjoint sets;
2. for each \(n\), \(\mathcal{V}_n\) refines \(\mathcal{U}_n\); and
3. \(\bigcup_{n<\infty} \mathcal{V}_n\) is a member of \(\mathcal{B}\).

With \(\mathcal{O}\) denoting the collection of all open covers of topological space \(X\), \(\mathcal{S}_{c}(\mathcal{O}, \mathcal{O})\) is selective screenability.

Addis and Gresham noted that countable dimensional metrizable spaces are selectively screenable and asked if the converse is true. Roman Pol, in [7], showed that the answer is no. We will now show that the countable dimensional metric spaces are exactly characterized by a game-theoretic version of selective screenability.

The following game, denoted \(G_c(\mathcal{A}, \mathcal{B})\), is naturally associated with \(\mathcal{S}_{c}(\mathcal{A}, \mathcal{B})\). Players ONE and TWO play as follows: In the \(n\)-th inning, ONE first chooses \(\mathcal{O}_n\), a member of \(\mathcal{A}\), and then TWO responds with \(\mathcal{T}_n\) which is pairwise disjoint and refines \(\mathcal{O}_n\). A play \((\mathcal{O}_1, \mathcal{T}_1, \ldots, \mathcal{O}_n, \mathcal{T}_n, \ldots)\) is won by TWO if \(\bigcup_{n<\infty} \mathcal{T}_n\) is a member of \(\mathcal{B}\); else, ONE wins. We can consider versions of different lengths of this game. For an ordinal number \(k\), let \(G^k_c(\mathcal{A}, \mathcal{B})\) be the game played as follows: In the \(l\)-th inning \((l < k)\), ONE first chooses \(\mathcal{O}_l\), a member of \(\mathcal{A}\), and then TWO responds with a pairwise disjoint \(\mathcal{T}_l\) which refines \(\mathcal{O}_l\). A play \(\mathcal{O}_0, \mathcal{T}_0, \ldots, \mathcal{O}_l, \mathcal{T}_l, \ldots l < k\) is won by TWO if \(\bigcup_{l<k} \mathcal{T}_l\) is a member of \(\mathcal{B}\); else, ONE wins. Thus, the game \(G_c(\mathcal{A}, \mathcal{B})\) is \(G^\omega_c(\mathcal{A}, \mathcal{B})\).

2. Main Results

From now on we assume that the spaces we work with are metrizable. We will see how selective screenability is related to covering dimension by showing that

1. a metrizable space is countable-dimensional if and only if TWO has a winning strategy in the game \(G^\omega_c(\mathcal{O}, \mathcal{O})\) (Theorem 2.2);
2. for each nonnegative integer \(n\), a metrizable space \(X\) is \(\leq n\)-dimensional if and only if TWO has a winning strategy in \(G^{n+1}_c(\mathcal{O}, \mathcal{O})\) (Theorem 2.4).
We will use the following result:

**Lemma 2.1** ([6, Theorem II.XI.21.2]). Let $X$ be a space and let $Y$ be a subspace of $X$. Let $(V_i : i \in I)$ be a collection of subsets of $Y$ open in $Y$. Then there is a collection $(U_i : i \in I)$ of open subsets of $X$ such that for each $i \in I$, we have $V_i = Y \cap U_i$, and for each finite subset $F$ of $I$, if $\cap_{i \in F} V_i = \emptyset$, then $\cap_{i \in F} U_i = \emptyset$.

**Theorem 2.2.** Let $X$ be a metric space.

1. If $X$ is countable dimensional, then TWO has a winning strategy in $G_\omega^{c(O, O)}$.
2. If TWO has a winning strategy in $G_\omega^{c(O, O)}$, then $X$ is countable dimensional.

**Proof of (1):** Let $X$ be countable dimensional, i.e., $X = \bigcup_{n < \infty} X_n$ where each $X_n$ is zero-dimensional. We will define a Markov strategy (for definition, see [4]) $\sigma$ for player TWO: For an open cover $U$ of $X$ and $n < \infty$, $U$ is an open cover of $X_n$. Since $X_n$ is zero-dimensional, find a pairwise disjoint family $V$ of subsets of $X_n$ open in $X_n$ such that $V$ covers $X_n$ and refines $U$. By Lemma 2.1, choose a pairwise disjoint family $\sigma(U, n)$ of open subsets of $X$ refining $U$ such that each element $V$ of $V$ is of the form $U \cap X_n$ for some $U \in \sigma(U, n)$.

Now TWO plays as follows: In inning 1, ONE plays $U_1$, and TWO responds with $\sigma(U_1, 1)$, thus covering $X_1$. When ONE has played $U_2$ in the second inning, TWO responds with $\sigma(U_2, 2)$, thus covering $X_2$, and so on. And in the $n$-th inning, when ONE has chosen the cover $U_n$ of $X$, TWO responds with $\sigma(U_n, n)$, covering $X_n$. This strategy evidently is a winning strategy for TWO.

**Proof of (2):** Let $\sigma$ be a winning strategy for TWO. Let $B$ be a base for the metric space $X$. For each $n$, let $B_n$ be the family $\{B \in B : \text{diam}(B) < \frac{1}{n}\}$. Consider the plays of the game in which, in each inning, ONE plays $U_1$, and TWO responds with $\sigma(U_1, 1)$, thus covering $X_1$. When ONE has played $U_2$ in the second inning, TWO responds with $\sigma(U_2, 2)$, thus covering $X_2$, and so on. And in the $n$-th inning, when ONE has chosen the cover $U_n$ of $X$, TWO responds with $\sigma(U_n, n)$, covering $X_n$. This strategy evidently is a winning strategy for TWO.

Define a family $(C_\tau : \tau \in <^\omega \mathbb{N})$ of subsets of $X$ as

1. $C_\emptyset = \cap \{\cup \sigma(B_n) : n < \infty\}$;
2. for $\tau = (n_1, \ldots, n_k)$, $C_\tau = \cap \{\cup \sigma(B_{n_1}, \ldots, B_{n_k}, B_n) : n < \infty\}$.

We will show that $X = \bigcup \{C_\tau : \tau \in <^\omega \mathbb{N}\}$. Suppose, to the contrary, that $x \notin \bigcup \{C_\tau : \tau \in <^\omega \mathbb{N}\}$. Let us choose an $n_1$ such that $x \notin \sigma(B_{n_1})$. With $n_1, \ldots, n_k$ chosen such that $x \notin \sigma(B_{n_1}, \ldots, B_{n_k})$,
let us choose an \( n_{k+1} \) such that \( x \notin \sigma(B_{n_1}, \ldots, B_{n_{k+1}}) \), and so on. Then
\[
B_{n_1}, \sigma(B_{n_1}), B_{n_2}, \sigma(B_{n_1}, B_{n_2}), \ldots
\]
is a \( \sigma \)-play lost by TWO, contradicting the fact that \( \sigma \) is a winning strategy for TWO.

Also, we will show that each \( C_\tau \) is zero-dimensional. Let \( x \in C_\tau \) and let \( \tau = (n_1, \ldots, n_k) \) be given. Thus, \( x \) is a member of \( \cap \{ \cup \sigma(B_{n_1}, \ldots, B_{n_k}, B_n) : n < \infty \} \). For each \( n \), choose a neighborhood \( V_n(x) \in \sigma(B_{n_1}, \ldots, B_{n_k}, B_n) \). Since for each \( n \) we have \( \text{diam}(V_n(x)) < \frac{1}{n} \), the set \( \{ V_n(x) \cap C_\tau : n < \infty \} \) is a neighborhood basis for \( x \) in \( C_\tau \). Also, we have that each \( V_n(x) \) is closed in \( C_\tau \) because of disjointness of TWO’s chosen sets. The set \( V = \cup \sigma(B_{n_1}, \ldots, B_{n_k}, B_n) \setminus V_n(x) \) is open in \( X \) and so \( C_\tau \setminus V_n(x) = C_\tau \setminus V \) is open in \( C_\tau \). Thus, each element of \( C_\tau \) has a basis consisting of clopen sets. □

Observe that in the proof of Theorem 2.2 we show:

**Corollary 2.3.** Let \( X \) be a metric space. The following are equivalent.

1. TWO has a winning strategy in \( G_\omega^\infty(O, O) \).
2. TWO has a winning Markov strategy in \( G_\omega^\infty(O, O) \).

The proof of the following theorem uses the ideas in the proof of Theorem 2.2.

**Theorem 2.4.** Let \( X \) be a metric space. The following are equivalent.

1. If \( X \) is \( \leq n \)-dimensional then TWO has a winning strategy in \( G_{\omega n+1}^\infty(O, O) \).
2. If TWO has a winning strategy in \( G_{\omega n+1}^\infty(O, O) \), then \( X \) is \( \leq n \)-dimensional.

From this theorem, we obtain that the metric space \( X \) is \( n \)-dimensional if and only if TWO has a winning strategy in \( G_{\omega n+1}^\infty(O, O) \) but not in \( G_{\omega}^\infty(O, O) \).

**References**


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