Numerical methods for hyperbolic PDEs

1. Use Taylor series to derive the modified equations for the constant coefficient scalar advection case for the four methods discussed in class. Hint: You will need to use the PDE \( q_t + u q_x = 0 \) to express \( q_{tt} \) in terms of \( q_{xx} \).

2. A general scalar conservation law has the form

\[
q_t + f(q)_x = 0
\]

A numerical method is said to be conservative if it exactly conserves the quantity \( q \) (which may be something like mass, momentum, energy, heat, and so on). In this case, our numerical solution has the following property:

\[
\sum_{i=1}^{M} Q_i^{n+1} = \sum_{i=1}^{M} Q_i^n + \text{fluxes at the boundary}
\]

Show that a numerical scheme written in the form

\[
Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x} \left( F^n_{i+1/2} - F^n_{i-1/2} \right), \quad i = 1, \ldots, M
\]  

(1)

is conservative, up to fluxes at the boundary. Here, we assume that \( F^n_{i-1/2} \) are numerical approximations to the flux function \( f(q) \) evaluated at cell edges \( x_{i-1/2} \). Don’t worry (yet) about how we might actually approximate the flux numerically.

3. The scalar constant coefficient advection equation we discussed in class is equivalent to the conservative form of the equation given by

\[
q_t + (u q)_x = 0
\]

Written in this way, the flux function is seen to be \( f(q) = u q \). If we want to evaluate this flux function at cell edges \( x_{i-1/2} \), we need to have a value for \( Q_{i-1/2}^n \). But all we have are cell averages \( Q_i^n \) and \( Q_{i-1}^n \). The question then arises, What is the best way to compute \( Q_{i-1/2}^n \) from these values?

(a) Show that simply setting \( Q_{i-1/2}^n \) equal to the average of \( Q_i^n \) and \( Q_{i-1}^n \) leads to the unstable FTCS scheme.

(b) For \( u > 0 \), show that for times \( t > t_n \), a reasonable value for \( q \) at the cell edge \( x_{i-1/2} \) is \( Q_{i-1/2}^n \), so that the flux at this edge is given by \( F_{i-1/2} = u Q_{i-1/2}^n \). Using this flux, show that the upwind scheme

\[
Q_{i}^{n+1} = Q_{i}^{n} - \frac{u \Delta t}{\Delta x} \left( Q_{i}^{n} - Q_{i-1}^{n} \right)
\]

can be written in the form of (1).

(c) How will the scheme differ for the case \( u < 0 \)? Write down a general expression for the numerical flux \( F_{i-1/2}^n \) that takes into account both cases \( u > 0 \) and \( u < 0 \).

4. Using the code test_hyper.m, determine the experimental order of convergence for the upwind scheme, and the Lax-Wendroff scheme (without limiters). As initial conditions, use the smooth solution provided in the code. Compute the 1-norm errors at time \( T = t_N = 1 \) using the formula

\[
e_1^M = \sum_{i=1}^{M} |q(x_i, 0) - Q_i^N| \Delta x.
\]

Determine the numerical order of convergence \( p \) by computing errors on grids \( M = 20, 40, 80, 160 \). You may use the assumption that

\[
e_1^M \approx C(\Delta x)^p, \quad \Delta x = \frac{1}{M}
\]
for some constant $C$ to determine the convergence rate $p$ between successive grids.
Show that the convergence rates you obtain agree with what we expect theoretically. Plot the errors for mesh size $\Delta x$ on a loglog plot.

5. From the modified PDE for the Lax-Wendroff method, one can see that the dispersive error term depends on the CFL number. Show this numerically by running the test_hyper.m code with the step function for CFL numbers in the range 0.01, 0.1, 0.5, 0.9. Produce several plots illustrating the effect of choice of CFL on the numerical solution obtained using Lax-Wendroff. Include the effect that the limiters have on this solution. Do the limiters completely suppress the spurious oscillations?

6. Using the solutions you obtained from test_hyper.m, verify that the numerical schemes we discussed in class are numerically conservative.

7. Construct a hyperbolic system of the form

$$q_t + Aq_x = 0$$

using a matrix $A \in \mathbb{R}^{2\times2}$ of your choosing. Using the lecture notes from class, solve the Riemann problem for this equation.

8. Using the solution you found in Problem 7, determine the solution $q(x_{i-1/2}, t)$ for $t > t_n$ from solution data $Q^n_i$ and $Q^{n-1}_i$. Use this solution at cell edge $x_{i-1/2}$ to write down an expression for the numerical flux function $F^{n}_{i-1/2}$ for this problem. Use this to write down a first order upwind scheme for this system.

9. Write a computer code in Matlab (using test_hyper.m as as starting point) that solves the system numerically using the upwind scheme. Start with piecewise constant initial data. Include your exact solution to show that your numerical solution agrees with the exact solution. Print out plots showing this agreement for several values of $t$. Note: For this system, your numerical solution $Q^n_i$ is actually a $2 \times 1$ vector. Show each component (along with the exact solution for that component) on separate plots.