

SPM BULLETIN

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1. EDITOR'S NOTE

Open problem solved. We are glad to announce the first solution of a *Problem of the Month* posed in the *SPM Bulletin*. The problem from the third issue was solved by Lubomyr Zdomsky, a student of Taras Banakh. This solution is a part of a large project carried by these two mathematicians, which will hopefully be announced in this bulletin when it is finished.

Past problems. We also have a new section on past problems in the *SPM Bulletin*.

Paper on open problems. We have finished writing a paper containing a significant portion of the important problems in the field of SPM. Email us to get a copy of the paper.

ArXiv papers. From now on, we will try to include announcements of papers which are of interest to readers of this bulletin and were recently announced in the *Mathematics ArXiv*.

We encourage all contributors to submit their papers to the Mathematics ArXiv prior to the submission of an announcement to the *SPM Bulletin*, so to guarantee

larger exposure for their papers. (The submission to the ArXiv does not guarantee the inclusion of the announcement in this bulletin, so please also email us a note concerning your submission to the ArXiv after you complete it.) Submissions to other e-print servers can also be considered, upon request.

Contributions to the next issue are, as always, welcome.

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2. RESEARCH ANNOUNCEMENTS

2.1. Models in which every nonmeager set is nonmeager in a nowhere dense Cantor set. We prove that it is relatively consistent with ZFC that in any perfect Polish space, for every nonmeager set A there exists a nowhere dense Cantor set C such that $A \cap C$ is nonmeager in C . We also examine variants of this result and establish a measure theoretic analog.

<http://arxiv.org/abs/math.LO/0311443>
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2.2. The γ -Borel conjecture. In this paper we show that it is relatively consistent with ZFC that every γ -set is countable while not every strong measure zero set is countable. This answers a question of Paul Szeptycki. A set is a γ -set iff every ω -cover contains a γ -subcover. An open cover is an ω -cover iff every finite set is covered by some element of the cover. An open cover is a γ -cover iff every element of the space is in all but finitely many elements of the cover. Gerlits and Nagy proved that every γ -set has strong measure zero. We also show that it is consistent that every strong γ -set is countable while there exists an uncountable γ -set. On the other hand every strong measure zero set is countable iff every set with the Rothberger property is countable.

<http://arxiv.org/abs/math.LO/0312308>
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2.3. Consistency of a counterexample to Naimark's problem. We construct a C^* -algebra that has only one irreducible representation up to unitary equivalence but is not isomorphic to the algebra of compact operators on any Hilbert space. This answers an old question of Naimark. Our construction uses a combinatorial statement called the diamond principle, which is known to be consistent with but not provable from the standard axioms of set theory (assuming those axioms are consistent). We prove that the statement "there exists a counterexample to Naimark's problem which is generated by \aleph_1 elements" is undecidable in standard set theory.

<http://arxiv.org/abs/math.OA/0312135>
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2.4. Comparing the automorphism group of the measure algebra with some groups related to the infinite permutation group of the natural numbers.

We prove, by a straight construction, that the automorphism group of the measure algebra and the subgroup of the measure preserving ones cannot be isomorphic to the trivial automorphisms of $P(\mathbb{N})/fin$.

<http://arxiv.org/abs/math.LO/0312472>

Pietro Ursino

2.5. Quantitative study of semi-Pfaffian sets. We study the topological complexity of sets defined using Khovanskii's Pfaffian functions, in terms of an appropriate notion of format for those sets. We consider semi- and sub-Pfaffian sets, but more generally any definable set in the o-minimal structure generated by the Pfaffian functions, using the construction of that structure via Gabrielov's notion of limit sets. All the results revolve around giving effective upper-bounds on the Betti numbers (for the singular homology) of those sets.

<http://arxiv.org/abs/math.AG/0401079>

Thierry Zell

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2.6. \mathfrak{o} -bounded groups and other topological groups with strong combinatorial properties. We construct several topological groups with very strong combinatorial properties. In particular, we give simple examples of subgroups of \mathbb{R} (thus strictly \mathfrak{o} -bounded) which have the Hurewicz property but are not σ -compact, and show that the product of two \mathfrak{o} -bounded subgroups of ${}^{\mathbb{N}}\mathbb{R}$ may fail to be \mathfrak{o} -bounded, even when they satisfy the stronger property $S_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$. This solves a problem of Tkačenko and Hernandez, and extends independent solutions of Krawczyk and Michalewski and of Banach, Nickolas, and Sanchis. We also construct separable metrizable groups G of size continuum such that every countable Borel ω -cover of G contains a γ -cover of G .

<http://arxiv.org/abs/math.GN/0307225>

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3. PROBLEM OF THE MONTH

Let us write $BC(P)$ for the Borel Conjecture for sets with property P , that is, the hypothesis that every set of reals with property P is countable.

In [§2.2 above] Miller proves that $BC(S_1(\mathcal{O}, \mathcal{O}))$ implies (and is therefore equivalent to) $BC(SMZ)$, where SMZ stands for *strong measure zero*. The proof splits into two cases: $\aleph_1 = \mathfrak{b}$ and $\aleph_1 < \mathfrak{b}$. In the case $\aleph_1 < \mathfrak{b}$ Miller really shows that strong measure

zero plus $\mathbf{U}_{fin}(\mathcal{O}, \Gamma)$ implies $\mathbf{S}_1(\mathcal{O}, \mathcal{O})$. In Theorems 14 and 19 of [4] it is shown that in fact,

$$SMZ + \mathbf{U}_{fin}(\mathcal{O}, \Gamma) \Leftrightarrow \mathbf{S}_1(\mathcal{O}, \mathcal{O}) + \mathbf{U}_{fin}(\mathcal{O}, \Gamma) \Leftrightarrow (*_{GN},$$

where $(*)_{GN}$ is the Gerlitz-Nagy covering property introduced in [2]. This characterization implies that $(*)_{GN}$ is strictly stronger than $\mathbf{S}_1(\mathcal{O}, \mathcal{O})$, since it implies that $\text{non}((*)_{GN}) = \min\{\text{non}(\mathbf{S}_1(\mathcal{O}, \mathcal{O})), \text{non}(\mathbf{U}_{fin}(\mathcal{O}, \Gamma))\} = \min\{\text{cov}(\mathcal{M}), \mathfrak{b}\} (= \text{add}(\mathcal{M}))$, and it is consistent that $\mathfrak{b} < \text{cov}(\mathcal{M})$.

So we have:

Theorem 3.1. $BC((*)_{GN}) \Leftrightarrow BC(SMZ)$.

Proof. Assume that $\aleph_1 < \mathfrak{b}$ and there exists an uncountable strong measure zero set X . As SMZ is hereditary, we may assume that $|X| = \aleph_1$, and therefore X satisfies $\mathbf{U}_{fin}(\mathcal{O}, \Gamma)$ as well, that is, X satisfies $(*)_{GN}$.

Next, assume that $\aleph_1 = \mathfrak{b}$.

Consider the collection Ω^{gp} of open ω -covers \mathcal{U} of X such that there exists a partition \mathcal{P} of \mathcal{U} into finite sets such that for each finite $F \subseteq X$ and all but finitely many $\mathcal{F} \in \mathcal{P}$, there exists $U \in \mathcal{F}$ such that $F \subseteq U$ [3].

In [3] it is proved that $\mathbf{S}_1(\Omega, \Omega^{gp})$ is equivalent to having $(*)_{GN}$ in all finite powers. Now if $\aleph_1 = \mathfrak{b}$ then by [1] there exists an uncountable element X in $\mathbf{S}_1(\Omega, \Omega^{gp})$ (in particular, X satisfies $(*)_{GN}$). \square

It is a conjecture of Tomasz Weiss that $(*)_{GN}$ is closed under taking finite products. If $(*)_{GN}$ is closed under taking finite powers, then $(*)_{GN} = \mathbf{S}_1(\Omega, \Omega^{gp})$ and we have that $BC(\mathbf{S}_1(\Omega, \Omega^{gp})) \Leftrightarrow BC(SMZ)$. Otherwise, I do not even know whether $BC(\mathbf{S}_1(\Omega, \Omega))$ implies $BC(SMZ)$ (it does if $(*)_{GN}$ implies $\mathbf{S}_1(\mathcal{O}, \mathcal{O})$ in all finite powers.) In fact we need to prove the following.

Conjecture 3.2. *If X has strong measure zero and $|X| < \mathfrak{b}$, then all finite powers of X have strong measure zero (equivalently, all finite powers of X satisfy $\mathbf{S}_1(\mathcal{O}, \mathcal{O})$).*

This constitutes the *Problem of the month*.

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4. PROBLEMS FROM EARLIER ISSUES

In this section we list the past problems posed in the *SPM Bulletin*. For definitions, motivation and related results, consult the corresponding issue.

For conciseness, we make the convention that all spaces in question are zero-dimensional, separable metrizable spaces.

Issue 1. *Is $\left(\frac{\Omega}{\Gamma}\right) = \left(\frac{\Omega}{\mathbb{T}}\right)$?*

Issue 2. *Is $\mathbf{U}_{fin}(\Gamma, \Omega) = \mathbf{S}_{fin}(\Gamma, \Omega)$? And if not, does $\mathbf{U}_{fin}(\Gamma, \Gamma)$ imply $\mathbf{S}_{fin}(\Gamma, \Omega)$?*

Issue 3. *Does there exist (in ZFC) a set satisfying $\mathbf{U}_{fin}(\mathcal{O}, \mathcal{O})$ but not $\mathbf{U}_{fin}(\mathcal{O}, \Gamma)$?*

Solution. **Yes** (Lubomyr Zdomsky, 2003). □

Issue 4. Does $S_1(\Omega, T)$ imply $U_{fin}(\Gamma, \Gamma)$?

Issue 5. Is $\mathfrak{p} = \mathfrak{p}^*$?

Issue 6. Does there exist (in ZFC) an uncountable set satisfying $S_1(\mathcal{B}_\Gamma, \mathcal{B})$?

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- [3] Lj. D. R. Kočinac and M. Scheepers, *Combinatorics of open covers (VII): groupability*, Fundamenta Mathematicae **179** (2003), 131–155.
- [4] A. Nowik, M. Scheepers, and T. Weiss, *The algebraic sum of sets of real numbers with strong measure zero sets*, J. Symbolic Logic **63** (1998), 301–324.

Previous issues. The first issues of this bulletin, which contain general information (first issue), basic definitions, research announcements, and open problems (all issues) are available online, on <http://arxiv.org/abs/math.GN/x>, where x is 0301011, 0302062, 0303057, 0304087, 0305367, and 0312140, respectively, for issues number 1 to 6.

Contributions. Please submit your contributions (announcements, discussions, and open problems) by e-mailing us. It is preferred to write them in \LaTeX . The authors are urged to use as standard notation as possible, or otherwise give the definitions or a reference to where the notation is explained. Contributions to this bulletin would not require any transfer of copyright, and material presented here can be published elsewhere.

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