MATH 301 – Final Examination

This take-home part of the exam is due at 1:00 p.m. on 14 May 2003.

1) You are encouraged to use Maple (or similar software) for this problem. In Maple, the symbol \( \infty \) is obtained by typing `infinity` and the symbol \( e^x \) is obtained by typing `exp(x)`. You may also encounter strange output. If so, use the command `evalf` to obtain a numerical approximation to the strangeness.

   The goal here is to find good polynomial approximations to
   \[
   \phi(x) = x^2 \sin x. \tag{1}
   \]

   a) Use Maple to graph \( \phi(x) \) on the interval \([-3,3]\).

   b) Each approximating polynomial is an element of the inner product space \( P_n \), where the inner product is given by
   \[
   \langle f, g \rangle = \int_{-\infty}^{\infty} e^{-x^2} f(x) g(x) \, dx. \tag{2}
   \]

   The function \( e^{-x^2} \) is known as a *weighting function*. Create an orthogonal basis for \( P_5 \), where orthogonality is determined in terms of the inner product (2). The elements in your orthogonal basis are the *Hermite polynomials*.

   c) For each \( j = 0, 1, 2, 3, 4, 5 \), determine the best least squares approximation \( p_j^* \) for (1), where each \( p_j^* \) is in the inner product space \( P_j \) with inner product (2). For each \( j \), plot (1) and \( p_j^* \) on the same set of axes.

   d) Plot, on the same set of axes, all of your polynomial approximations \( p_j^* \) as well as the function (1). Write a brief essay about your observations with respect to this picture and this entire take-home exercise.