Some Games Related to Perfect Spaces

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Abstract

We define some new topological games motivated by covering axioms related to countable paracompactness. We investigate the basic relationships between existence of winning strategies in these games and we establish interesting connections between these games and the Banach Mazur game, generalized metric spaces, perfect spaces Michael type lines and Bing’s G.

Introduction. Let \( X \) be a \( T_1 \)-space. Recall the well known characterization of countable paracompactness and countable metacompactness: a space \( X \) is countably paracompact (respectively countably metacompact) if and only if for every descending sequence

\[
C_0 \supset C_1 \supset \ldots \supset C_n \supset \ldots
\]

of closed sets such that \( \bigcap C_n = \emptyset \), there are open sets \( V_n \supset C_n \) for each \( n \in \omega \) such that \( \bigcap_n V_n = \emptyset \) (respectively \( \bigcap_n V_n = \emptyset \)).

A natural game, which we call the countably paracompact game on \( X \) and denote \( \text{CP}(X) \), is defined as follows: Players ONE and TWO play an inning for each positive integer. In the \( n \)-th inning ONE chooses a closed set \( C_n \) and TWO responds with an open set \( V_n \supset C_n \). ONE must further obey the rule that \( C_{n+1} \subset C_n \) for each integer \( n \). ONE wins the play of the game

\[
(C_0, V_0, \ldots C_n, V_n, \ldots)
\]

if \( \bigcap_n C_n = \emptyset \) and \( \bigcap_n V_n \neq \emptyset \). Clearly if \( X \) is not countably paracompact, then ONE has a winning strategy in \( \text{CP}(X) \), and if TWO has a winning strategy in \( \text{CP}(X) \), then \( X \) is countably paracompact.

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The *countably metacompact game*, \(CM(X)\), is defined as follows: ONE chooses closed sets \(C_n\) and TWO responds with open sets \(V_n \supset C_n\). ONE must again obey the rule that \(C_{n+1} \subset C_n\). ONE wins the play of the game

\[(C_0, V_0, ... C_n, V_n, ...)\]

if \(\bigcap_n C_n = \emptyset\) and \(\bigcap V_n \neq \emptyset\).

As was true for \(CP(X)\), if \(X\) fails to be countably metacompact then ONE has a winning strategy in the game \(CM(X)\), and if TWO has a winning strategy in \(CM(X)\), then \(X\) is countably metacompact. We will refer to the games \(CP(X)\) and \(CM(X)\) collectively as *descending sequence games*.

E.K. van Douwen introduced the class of paranormal spaces as a natural generalization of both normality and countable paracompactness. A space is said to be *paranormal* if for every countable discrete family \(\{D_n : n \in \omega\}\) of closed sets, there is a locally finite family \(\{U_n : n \in \omega\}\) of open sets such that \(U_n \supset D_n\) for each \(n \in \omega\). In the case that \(X\) is scattered of height two, \(X\) is countably paracompact if and only if \(X\) is paranormal.

There is a natural game associated to this property which we denote \(P_0(X)\): ONE chooses closed sets \(C_n\) and TWO responds with open sets \(V_n \supset C_n\). ONE must also obey the rule that \(C_m \cap C_n = \emptyset\) for each \(m \neq n\). ONE wins the play of the game

\[(C_0, V_0, ... C_n, V_n, ...)\]

if \(\{C_n : n \in \omega\}\) is discrete and \(\{V_n : n \in \omega\}\) is not locally finite.

*Metanormal* is the metacompact analogue to paranormal. The definition is the same as for paranormal except one requires the family of open sets to be point-finite instead of locally finite. Every countably metacompact space is indeed metanormal and for scattered spaces of height two the properties coincide. The game analogue, denoted \(P_1(X)\), is defined as follows: ONE chooses closed sets \(C_n\) and TWO responds with open sets \(V_n \supset C_n\). ONE must also obey the rule that \(C_m \cap C_n = \emptyset\) for each \(m \neq n\). ONE wins the play of the game

\[(C_0, V_0, ... C_n, V_n, ...)\]

if \(\{C_n : n \in \omega\}\) is discrete and \(\{V_n : n \in \omega\}\) is not point finite.

We will refer to the games \(P_0(X)\) and \(P_1(X)\) as the *partition games*.

The game \(CM(X)\) is due to Telgarsky [19] who proved that \(X\) is a Morita P-space if and only if TWO has a winning strategy in \(CM(X)\). The other
games are new. We refer the reader to [18] for the basic results and definitions related to topological games.

**Winning strategies.** The following relationship between the games follow directly from the definitions.

**Theorem 1** Let $X$ be a $T_1$ space.

1. If ONE has a winning strategy in $\text{CM}(X)$ then ONE has a winning strategy in $\text{CP}(X)$.

2. If ONE has a winning strategy in $P_1(X)$ then ONE has a winning strategy in $P_0(X)$.

3. If TWO has a winning strategy in $\text{CP}(X)$ then TWO has a winning strategy in $\text{CM}(X)$.

4. If TWO has a winning strategy in $P_0(X)$ then TWO has a winning strategy in $P_1(X)$.

None of the implications in the previous theorem can be reversed. This can be seen by considering the Mrówka-Isbell $\Psi$-space. If $A$ is a maximal almost disjoint family on $\omega$ let $\Psi(A)$ denote the topological space whose underlying set is $A \cup \omega$, and whose topology is determined by letting the points of $\omega$ be isolated, and for each $a \in A$, letting sets of the form $\{a\} \cup (a \setminus F)$ for $F$ a finite subset of $\omega$ constitute a neighborhood base at $a \in A$. See [5] for more on these spaces. $\Psi(A)$ is always Tychonoff, pseudocompact, but it is neither countably paracompact nor paranormal. But since it is of scattered height two and has only countably many isolated points, it is countably metacompact. Moreover, in a very strong sense none of the implications of Theorem 1 can be reversed:

**Theorem 2** If $A$ is a maximal almost disjoint family, then

1. ONE has a winning strategy in $\text{CP}(\Psi(A))$ and $P_0(\Psi(A))$, and

2. TWO has a winning strategy in $\text{CM}(\Psi(A))$ and $P_1(\Psi(A))$.

**Proof:** Since $\Psi(A)$ is not countably paracompact and not paranormal, let $\{C_n : n \in \omega\}$ be the sequence of closed sets witnessing the failure of either property. The strategy in which ONE plays $C_n$ in $n^{th}$ inning is a winning strategy for ONE. Note that this strategy only depends on the inning of the game. For the other clause we have a more general result:
Lemma 3 If $Y$ is scattered of height two and if $Y$ has only countably many isolated points, then TWO has a winning strategy in the games $\text{CM}(Y)$ and $P_1(Y)$.

Proof: Without loss of generality, $\omega$ is the set of isolated points of $Y$. If $D_n$ is ONE’s play in the $n^{th}$ inning of either game, let TWO play $U_n = \sigma(D_1, ..., D_n) = D_n \cup (\omega \setminus n)$. If $\bigcap_{n \in \omega} D_n = \emptyset$ then $\bigcap_{n \in \omega} U_n = \emptyset$, therefore this strategy is winning for player TWO. Note that this strategy only depends only on the inning of the game and the last play of player ONE.

If $X$ is normal then countable metacompactness is equivalent to countable paracompactness, and $X$ is both paranormal and metanormal. A similar relationship holds for the descending sequence games:

Theorem 4 Suppose that $X$ is normal. Then TWO has a winning strategy in $\text{CM}(X)$ if and only if TWO has a winning strategy in $\text{CP}(X)$.

Proof: If $X$ is normal, then for each closed set $F$ and each open set $U$ there is an open set $V(F, U)$ such that

$$F \subset V(F, U) \subset \overline{V(F, U)} \subset U.$$ 

If $\sigma$ is a winning strategy for TWO in the game $\text{CM}(X)$, then $\sigma'$ defined by

$$\sigma'(C_0, U_0, ..., C_n) = V(C_n, \sigma(C_0, U_0, ..., C_n)).$$

is a winning a winning strategy for TWO in $\text{CP}(X)$.

Interestingly, the analogue to Theorem 4 does not hold true for the partition games. This follows from the next theorem and Theorem 8 below.

Theorem 5 Suppose that $X$ is a normal space of scattered height two. Then TWO has a winning strategy in $P_1(X)$.

Proof: Let $X = D \cup I$ where $I$ is the set of isolated points in $X$ and $D$ is closed discrete in $X$. We describe a winning strategy $\sigma$ for TWO. Let $C_0 \subseteq X$ be closed. Fix $V_0 \subseteq X$ open such that $C_0 \cap D \subseteq V_0$ and such that $\overline{V_0} \cap D = C_0 \cap D$. Then TWO plays $U_0 = \sigma(C_0) = V_0 \cup (C_0 \cap I)$. Suppose that $C_0, U_0, ..., C_n, U_n$ is a partial play of the game and suppose that $U_k = V_k \cup (C_k \cap I)$ for each $k \leq n$ where
1. \( V_k \cap V_j = \emptyset \) for each \( j < k \leq n \), and
2. \( V_k \cap D = C_k \cap D \) for each \( j < k \leq n \).

Suppose that ONE plays \( C_{n+1} \) disjoint from each \( C_k \). Fix \( V_{n+1} \) open such that
1. \( V_{n+1} \cap D = V_{n+1} \cap D = C_{n+1} \cap D \), and
2. \( V_{n+1} \cap \bigcup_{k \leq n} V_k = \emptyset \).

Then TWO plays the open set

\[ U_{n+1} = \sigma(C_0, U_0, \ldots, C_n, U_n, C_{n+1}) = V_{n+1} \cup (C_{n+1} \cap I) . \]

It is easy to verify that this is a winning strategy for TWO.

We can not expect that winning strategies for TWO in any of the games implies that \( X \) is normal. Indeed if \( X \) is countably compact then TWO has a trivial winning strategy in all of the games. And there are countably compact not normal spaces (e.g., \( \omega_1 \times (\omega_1 + 1) \)). However, we have no example of a non-normal space of scattered height two for which TWO has a winning strategy in either \( CP(X) \) or \( P_0(X) \).

**Question 1** Assume \( X \) is of scattered height two and that TWO has a winning strategy in \( CP(X) \) or \( P_0(X) \). Is \( X \) normal?

**Bing’s space.** Other than the theorems thus far mentioned, there are no other implications between existence of winning strategies in the partition or descending sequence games even for the class of normal spaces. This can be established by considering the games on Bing’s space. Let \( \kappa \) be an uncountable cardinal and for each \( \alpha < \kappa \) let \( \chi_\alpha \in \{0, 1\}^{\mathcal{P}(\kappa)} \) be defined by

\[ \chi_\alpha(y) = 1 \text{ iff } \alpha \in y. \]

Let \( M = \{\chi_\alpha : \alpha \in \kappa\} \). Notice that as a subspace of \( \{0, 1\}^{\mathcal{P}(\kappa)} \) with the product topology \( M \) is discrete. Bing’s \( G \) is \( G(\kappa) = \{0, 1\}^{\mathcal{P}(\kappa)} \) with the topology obtained by isolating all the points of \( G(\kappa) \setminus M \). Therefore a basic open neighborhood of \( \chi_\alpha \in M \) is of the form

\[ U_F(\chi_\alpha) = \{\chi_\alpha\} \cup \{f \in G(\kappa) \setminus M : \forall y \in F f(y) = 1 \text{ iff } \alpha \in y\} . \]
where $F \subseteq \mathcal{P}(\kappa)$ is finite.

It is well known that $G(\kappa)$ is collectionwise normal and of scattered height two. Therefore by Theorem 5, TWO has a winning strategy in the game $P_1(X)$.

**Theorem 6** Let $\kappa$ be an uncountable cardinal. ONE has a winning strategy in $\text{CM}(G(\kappa))$ (hence in $\text{CP}(G(\kappa))$).

**Proof:** We describe a strategy $\sigma$ for ONE in the game $\text{CM}(G(\kappa))$. ONE opens the game by playing $\sigma(\emptyset) = C_0 = M$. Suppose that $(C_0, U_0, \ldots, C_n, U_n)$ is a partial play of the game such that $C_k$ is uncountable for each $k \leq n$. For each $\alpha \in \kappa$ such that $\chi_\alpha \in C_n$ let $F^n_\alpha \subseteq \mathcal{P}(\kappa)$ be a finite set such that $U_{F^n_\alpha}(\chi_\alpha) \subseteq U_n$. Using the $\Delta$-system lemma, fix $C_{n+1} \subset C_n$ uncountable, $F_{n+1}$ finite, and $s_{n+1} : F_{n+1} \to 2$ such that

1. $\{F^n_\alpha : \chi_\alpha \in C_{n+1}\}$ forms a $\Delta$-system with root $F_{n+1}$ and
2. $\chi_\alpha F_{n+1} = s_{n+1}$ for all $\alpha \in C_{n+1}$.

Let $\sigma(C_0, U_0, \ldots, C_n, U_n) = C_{n+1}$.

Now, suppose that $(C_0, U_0, \ldots, C_n, U_n, \ldots)$ is a play of the game where ONE follows the strategy $\sigma$.

**Proposition 7** $\bigcap_{n \in \omega} U_n \neq \emptyset$.

**Proof:** Since $m < n$ implies $C_n \subset C_m$ we have that if $y \in F_m \cap F_n$, then $s_m(y) = s_n(y)$. Let $F = \bigcup_{n \in \omega} F_n$. Choose $\alpha_n \in \kappa$ recursively so that

1. $\alpha_n \in C_n$ for each $n \in \omega$.
2. $(F^n_{\alpha_n} \setminus F_n) \cap F = \emptyset$ for each $n \in \omega$.
3. $(F^n_{\alpha_n} \setminus F_n) \cap (F^m_{\alpha_m} \setminus F_m) = \emptyset$ for each $m \neq n$.

Therefore $\chi_{\alpha_n}$ and $\chi_{\alpha_m}$ agree on $F^n_{\alpha_n} \cap F^m_{\alpha_m}$, so there is an $f \in G(\kappa)$ such that $fF^n_{\alpha_n} = \chi_{\alpha_n}F^n_{\alpha_n}$ for each $n \in \omega$. Therefore $f \in \bigcap_{n \in \omega} U_n$.

**Remark:** Let $\kappa \geq (2^\omega)^+$ and let $X$ denote $G(\kappa)$ endowed with the $G_\delta$ topology. So a typical basic open set is of the form

$$U_F(\chi_\alpha) = \{\chi_\alpha\} \cup \{f \in X \setminus M : \forall y \in F \ f(y) = 1 \text{ iff } \alpha \in y\}$$
where $F \subseteq \mathcal{P}(\kappa)$ is countable. Then ONE has a winning strategy in $\mathcal{CM}(X)$ and Theorem 5 can be strengthened to player TWO has a winning strategy in $P_0(X)$. We need $\kappa \geq (2^\omega)^+$ to apply the $\Delta$-system lemma for countable subsets of $\kappa$ in order to prove that ONE has a winning strategy in $\mathcal{CM}(X)$. To prove that TWO has a winning strategy in $P_0(X)$, note that if $\{U_n : n \in \omega\}$ is the sequence of open sets generated by TWO’s strategy described in Theorem 5, then not only are the $U_n$’s pairwise disjoint, but having the $G_\delta$-topology guarantees that $\bigcup U_n = \bigcup U_n$. Therefore the family is not only locally finite but discrete. This implies that the described strategy is winning in the game $P_0(X)$.

However, TWO does not have a winning strategy in $P_0(G(\kappa))$.

**Theorem 8** Let $\kappa$ be an uncountable cardinal. TWO does not have a winning strategy in $P_0(G(\kappa))$.

**Proof:** Suppose that $\sigma$ is a strategy for TWO. Fix a countable elementary submodel $\mathcal{M}$ of $H_\theta$ where $\theta$ is sufficiently large so that $k, \mathcal{P}(\kappa), \sigma \in \mathcal{M}$. We use $\mathcal{M}$ to show how ONE should play to defeat $\sigma$. Let $\gamma \in \kappa \setminus \mathcal{M}$ be arbitrary. Also let $\alpha_0 \in \mathcal{M}$ be arbitrary and let ONE open with $\{\chi_{\alpha_0}\}$. Let $F_0 \subseteq \mathcal{P}(\kappa)$ be such that $U_{F_0}(\chi_{\alpha_0}) \subset \sigma(\{\chi_{\alpha_0}\})$. Let

$$X_0 = \bigcap\{y \in F_0 : \gamma \in y\} \setminus \bigcup\{y \in F_0 : \gamma \notin y\}.$$ 

Then since $X_0 \in \mathcal{M}$ but is not a subset, $X_0$ is uncountable. ONE then responds in the second inning by playing $\{\chi_{\alpha_1}\}$ where $\alpha_1 \in X_0 \cap \mathcal{M}$. We continue in this manner so that we have the play of the game

$$(\{\chi_{\alpha_0}\}, U_{F_0}(\chi_{\alpha_0}), \ldots, \{\chi_{\alpha_n}\}, U_{F_n}(\chi_{\alpha_n}), \ldots)$$

and sets $X_n \in \mathcal{M}$ where

$$X_n = \bigcap\{y \in F_i : \gamma \in y \text{ and } i \leq n\} \setminus \bigcup\{y \in F_n : \gamma \notin y \text{ and } i \leq n\}$$

and $\alpha_{n+1} \in X_n \cap \mathcal{M}$. Then $\gamma \in X_n$ for each $n$ and $\alpha_m \in X_n$ for each $m > n$. Therefore $\chi_{\gamma}(y) = \chi_{\alpha_m}(y)$ whenever $y \in F_n$ and $m > n$. We now claim that $\chi_{\gamma} \in \bigcup_n U_{F_n}(\chi_{\alpha_n})$. To see this, fix $F \subseteq \mathcal{P}(\kappa)$ and consider the basic open set $U_F(\chi_{\gamma})$. Choose $n$ large enough so that $(F \cap F_m) \subseteq \bigcup_{i < n} F_i$ for each $m > n$. Then, $\chi_{\gamma}(y) = \chi_{\alpha_m}(y)$ for each $y \in F_m$ and each $m > n$. Therefore $U_F(\chi_{\gamma}) \cap U_{F_m}(\chi_{\alpha_m}) \neq \emptyset$.

If $\kappa$ is measurable we get the following stronger result:
Theorem 9. Suppose that $\kappa$ is measurable. Then ONE has a winning strategy in $P_0(G(\kappa))$. Moreover ONE has a winning strategy playing only singletons.

Proof: Let $\mathcal{U}$ be a countably complete ultrafilter on $\kappa$. We describe a strategy for player ONE. Let $\alpha_0 \in \kappa$ and let $\{\chi_{\alpha_0}\}$ be ONE’s opening play of the game. We use the countably complete ultrafilter to modify the previous proof into a winning strategy for ONE. Without loss of generality, we may assume that TWO plays basic open sets. If in the $n^{th}$ inning we have the partial play of the game $$(\{\chi_{\alpha_0}\}, U_{F_0}(\chi_{\alpha_0}), \ldots, \{\chi_{\alpha_n}\}, U_{F_n}(\chi_{\alpha_n}))$$ Without loss of generality we may also assume that $U_{F_i}(\chi_{\alpha_0}) \cap M = \{\chi_{\alpha_i}\}$ for each $i < n$. Let $L = \{y \in \bigcup_{i<n} F_i : y \in \mathcal{U}\}$ and let $S = \{y \in \bigcup_{i<n} F_i : y \not\in \mathcal{U}\}$. Since $\mathcal{U}$ is an ultrafilter we may choose $\alpha_n \in X_n = (\bigcap L) \setminus (\bigcup S)$. ONE then plays $\{\chi_{\alpha_n}\}$ in the next inning.

To see that this strategy is winning, suppose that $$(\{\chi_{\alpha_0}\}, U_{F_0}(\chi_{\alpha_0}), \ldots, \{\chi_{\alpha_n}\}, U_{F_n}(\chi_{\alpha_n}), \ldots)$$ is a play of the game according to the described strategy. Let $U_n = U_{F_n}(\chi_{\alpha_n})$ and for each $n$ let $X_n$ be as defined above. Then $X_n \in \mathcal{U}$ so $X_\omega = \bigcap_n X_n \neq \emptyset$. We will show that for each $\alpha \in X_\omega$, $\alpha \in \bigcup_n U_n$ and therefore that ONE wins the play of the game. Let $\alpha \in X_\omega$ and fix a basic open set $U_F(\chi_{\alpha})$ around $\chi_{\alpha}$. For each $y \in F$ fix, if possible, $n_y$ to be the minimal $n \in \omega$ so that $y \in F_n$. If $n_y$ exists and $m > n_y$ then by choice of $\alpha$ and $\alpha_m$, $\chi_{\alpha_m}(y) = \chi_{\alpha}(y)$. Therefore if $m > n_y$ for each $y \in F$ for which $n_y$ exists, then $U_F(\alpha) \cap U_{F_m}(\chi_{\alpha_m}) \neq \emptyset$.

Question 2 Does ONE have a winning strategy in $P_0(G(\omega_1))$?

We also do not know whether any of the other implications between the existence of winning strategies in the descending sequence games and the existence of winning strategies in the partition games:

Question 3 Suppose $X$ is a Tychonoff space.

1. If TWO has a winning strategy in $CP(X)$ (respectively $CM(X)$) does TWO have a winning strategy in $P_0(X)$ (respectively $P_1(X)$)?
2. If ONE has a winning strategy in $P_1(X)$ does ONE have a winning strategy in $CM(X)$?

**The Michael Line.** Bing’s space $G$ is collectionwise normal and countably paracompact (and therefore countably metacompact, paranormal and metanormal). Since ONE has a winning strategy in both $CM(G)$ and $CP(G)$ both games have much less to do with countable paracompactness or countable metacompactness than might be expected. By considering the games on the Michael line we will see that the same is also true for the partition games. Furthermore this consideration will lead us to consistent examples for which all the games are undetermined. On the way we will establish a connection with the Banach-Mazur Game.

Let $X$ be a be a space and let $Y \subseteq X$. Let $X_Y$ denote the space whose topology is a refinement of the topology on $X$ obtained by isolating all the points of $X \setminus Y$. So $R_Q$ denotes the usual Michael line (we thank Winfried Just for conversations regarding the games on $R_Q$).

The Banach-Mazur game on a space $X$ – denoted $BM(X)$– is played as follows: ONE and TWO alternately choose the terms $W_1, V_1, \ldots, W_n, V_n, \ldots$ of a descending sequence of open subsets of $X$. In the $n$–th inning ONE first chooses $W_n$, and TWO responds with $V_n$. Such a play is won by TWO if the intersection of all selected sets is nonempty; ONE wins otherwise. Other versions of the game require one or both players to play a closure decreasing sequence of open sets. The existence of a winning strategy for either player in the restricted versions of the game is equivalent to the existence of a winning strategy for the player in the original version of the game. The best known theorem about this game appears in [12]:

**Theorem 10 (Banach-Mazur-Oxtoby)** ONE has a winning strategy in $BM(X)$ if and only if $X$ is not a Baire space.

In the theorems below we use: If $Z \subseteq Y$ and $V$ is an $X_Y$–open subset of $X$ such that $Z \subseteq V$, then there is an $X$–open subset $U$ of $X$ such that $Z \subseteq U \subseteq V$.

**Theorem 11** If $X$ is a Baire space and if $Y$ is a dense first category $F_\sigma$–subset of $X$, then TWO does not have a winning strategy in either $CM(X_Y)$ or in $P_1(X_Y)$ (hence TWO does not have a winning strategy in either $CP(X_Y)$ or $P_0(X_Y)$).
**Proof**: The last parenthetical remark follows from Theorem 1. Consider first the game \(CM(X,Y)\), let \(X\) and \(Y\) be as in the hypotheses, and let \(F\) be a strategy for TWO in the game \(CM(X,Y)\). Also, represent \(Y\) as \(\cup_{n<\infty}C_n\), where each \(C_n\) is a closed, nowhere dense subset of \(X\) such that \(C_n \subseteq C_{n+1}\). Notice that for every subset \(Z\) of \(Y\) and for every \(X\)-open set \(U\) with \(Z \subseteq U\), \(Z \subseteq \text{int}_X(U)\).

If \(F\) is a strategy for TWO in \(CM(X,Y)\) then we define a strategy \(G\) for ONE in \(BM(X)\) by simultaneously defining a related play of the game \(CM(X,Y)\) as follows: Let \(G(\emptyset) = X\) be ONE’s first move in the game \(BM(X)\). Let \(O_1 = Y\) be player ONE’s first move in \(CM(X,Y)\), and let TWO’s response in \(CM(X,Y)\) be given by \(U_1 = F(O_1)\). Given \((W_1, V_1, W_2, V_2, ..., W_n, V_n)\), a partial play of the game \(BM(X)\), and given \((O_1, U_1, ..., O_n, U_n)\), a partial play of the game \(CM(X,Y)\) we proceed as follows:

(a) Let \(W_{n+1} = G(V_1, V_2, ..., V_n) = V_n \cap \text{int}_X U_n \setminus C_n\).

Let \(V_{n+1}\) be TWO’s response in \(BM(X)\). We may assume that

(b) \(V_{n+1} \subseteq W_{n+1}\).

We next show how to extend the play of the game \(CM(X,Y)\). Let

(c) \(O_{n+1} = O_n \cap \overline{V_{n+1}}\), and

(d) \(U_{n+1} = F(O_1, ..., O_{n+1})\).

This completes the recursive definition of the strategy \(G\). By assumption and Theorem 10, \(G\) is not a winning strategy, so let \((W_1, V_1, W_2, V_2, ...)\) be a play of the game \(BM(X)\) where ONE loses. Let \((O_1, U_1, O_2, U_2, ...)\) be the corresponding dual play in the game \(CM(X,Y)\). Then by (d) this is a play of the game where TWO follows the strategy \(F\). The definition of \(W_n\) (clause (a)) implies that \(W_n \cap C_n = \emptyset\), therefore \(\overline{V_n} \cap C_n = \emptyset\). Therefore by (c) we have \(O_n \subseteq \overline{V_n} \cap Y\), so

(e) \(\bigcap_{n \in \omega} O_n = \emptyset\).

Furthermore (a) also implies

(f) \(W_n \subseteq U_n\) for all \(n \in \omega\).
Finally, as \((W_1, V_1, W_2, V_2, \ldots)\) is a play of the game \(BM(X)\) where ONE loses, we have that \(\cap_{n \in \omega} W_n \neq \emptyset\). Therefore \(\cap_{n \in \omega} U_n \neq \emptyset\) by (f). This together with (e) implies that TWO loses the play \((O_1, U_1, O_2, U_2, \ldots)\) in the game \(CM(X_Y)\), hence \(F\) is not a winning strategy.

Now consider the game \(P_1(X_Y)\) and suppose that \(\sigma\) is a winning strategy for player TWO. This determines a strategy \(\tau = \tau_\sigma\) for player ONE in the game \(BM(X)\) as follows. Fix \(q_0 \in Y \setminus C_0\) and let \(U_0 = \sigma(q_0)\). Fix \(U'_0 \subseteq U_0\) such that \(\overline{U'_0} \cap C_0 = \emptyset\) and let \(\tau(\emptyset) = U'_0\) be ONE’s opening play in the game \(BM(X)\). Suppose in general that \((U'_0, V_0, \ldots, U'_n, V_n)\) is a play in \(BM(X)\) and that we have defined a corresponding play \(\{q_0\}, U_0, \ldots \{q_n\}, U_n\) in \(P_1(X_Y)\) where TWO follows the strategy \(\sigma\). Let \(q_{n+1} \in V_n \setminus (C_{n+1} \cup \{q_0, \ldots, q_n\})\) and let \(U_{n+1} = \sigma(\{q_0\}, \ldots, \{q_{n+1}\})\). Finally let \(U'_{n+1}\) be a nonempty open set such that \(\overline{U'_{n+1}} \subseteq U_{n+1} \cap V_n \setminus C_{n+1}\) and define

\[
\tau((U'_0, V_0, \ldots, U'_n, V_n)) = U'_{n+1}.
\]

Since \(\sigma\) is winning for TWO in \(P_1(X_Y)\) we have that \(\{U_n : n \in \omega\}\) is point finite, so \(\cap_{n \in \omega} U_n = \emptyset\). Then \(\cap_{n \in \omega} U'_n = \emptyset\) since \(U'_n \subseteq U_n\) for each \(n \in \omega\). Therefore \(\tau\) is a winning strategy for ONE in \(BM(X)\) contradicting that \(X\) is a Baire–space.

**Corollary 12** If \(X \subseteq \mathbb{R}\) is a Baire–space and if \(Y\) is a countable dense subset of \(X\), then TWO does not have a winning strategy in any of the games \(CM(X_Y), P_1(X_Y), CP(X_Y)\) and \(P_0(X_Y)\).

**Theorem 13** If \(X\) is a regular space such that TWO has a winning strategy in \(BM(X)\), and if \(Y\) is a dense first category \(F_\sigma\)–subset of \(X\), then ONE has a winning strategy in all of the games \(CM(X_Y), P_1(X_Y), CP(X_Y)\) and \(P_0(X_Y)\).

**Proof** : By Theorem 1 it suffices to prove that ONE has a winning strategy in \(CM(X_Y)\) and \(P_1(X_Y)\). We consider the case for \(CM(X_Y)\). Let \(C_n\) be as in the proof of Theorem 11, and let \(G\) be a winning strategy for TWO of \(BM(X)\). Since \(X\) is regular, for each nonempty open set \(U\), let \(\Phi(U)\) be a nonempty open set with \(\overline{\Phi(U)} \subseteq U\). We define a strategy \(F\) for ONE of \(CM(X_Y)\) by simultaneously playing \(BM(X)\):

In \(CM(X_Y)\), ONE’s first move is \(F(\emptyset) = O_1 = Y\). If TWO responds with \(U_1\), open in \(X_Y\). As in the previous proof, by taking its interior in \(X\) we may assume that \(U_1\) is open in \(X\).
Let $W_1 = \Phi(U_1 \setminus C_1)$ and let $V_1 = G(W_1)$. Therefore, $(W_1, V_1)$ is an inning of the game $BM(X)$ where TWO plays according to its strategy $G$.

In general, suppose that

$$(O_1, U_1, \ldots O_n, U_n)$$

and

$$(W_1, V_1, \ldots W_n, V_n)$$

are plays of the games $CM(X_Y)$ and $BM(X)$ respectively such that

1. $O_i \subseteq Y$ and $U_i$ is open in $X$ for each $i \leq n$.
2. $V_i = G(W_1, \ldots, W_i)$ for each $i \leq n$. So TWO plays according to the strategy $G$.
3. $W_i = \Phi((U_i \cap V_{i-1}) \setminus C_i)$ for each $i \leq n$. So $W_i \subseteq (U_i \cap V_{i-1}) \setminus C_i$.
4. $O_i = O_{i-1} \cap W_i$ for each $i \leq n$.

Then define $F((O_1, U_1, \ldots O_n, U_n) = O_n \cap \overline{W_n}$.

This defines a strategy for player ONE in the game $CM(X_Y)$. Suppose that $(O_1, U_1, \ldots O_n, U_n, \ldots)$ is a play of the game where ONE follows the strategy $F$ and let $(W_1, V_1, \ldots W_n, V_n, \ldots)$ be the corresponding play in the game $BM(X)$. By (3) and (4) $(\bigcap_{n \in \omega} O_n) \cap (\bigcup_{n \in \omega} C_n) = \emptyset$. Also by (2), this is a play where TWO follows the strategy $G$ so $\bigcap_{n \in \omega} V_n \neq \emptyset$. But then also $\bigcap_{n \in \omega} \overline{W_n} \neq \emptyset$, and by (3) we have that $\bigcap_{n \in \omega} U_n \neq \emptyset$. Therefore ONE wins this play of the game, so $F$ is a winning strategy.

Now we consider the game $P_1(X_Y)$: Suppose that $\tau$ is a winning strategy for TWO in the game $BM(X)$. We define a strategy $\sigma$ for ONE in $P_1(X_Y)$ as follows. Let $q_0 \in Y$ be arbitrary and let $\sigma(\emptyset) = \{q_0\}$. Suppose that TWO’s response is the set $U_0$. By taking its interior in $X$ we may assume that $U_0$ is open in $X$. Now consider the game $BM(X)$. Let $W_0 = U_0$ be ONE’s opening move, and suppose that $V_0 = \tau(W_0)$. We continue in this manner: suppose that the play $(\{q_0\}, U_0, \ldots, \{q_n\}, U_n)$ in $P_1(X_Y)$ and the play $(W_0, V_0, \ldots, W_n, V_n)$ in the game $BM(X)$ have been defined, so that

1. $W_i = U_i \cap V_{i-1}$ for each $i \leq n$.
2. $V_i = \tau(W_0, \ldots, W_i)$ for each $i \leq n$.
3. $\overline{V_i} \subseteq W_i$ for each $i \leq n$. 

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Then fix \( q_{n+1} \in V_n \setminus C_{n+1} \cup \{q_0, ..., q_n\} \) and define
\[
\sigma(U_0, ..., U_n) = \{q_{n+1}\}.
\]
We claim that this strategy is winning for ONE. So suppose that
\[
(\{q_0\}, U_0, ..., \{q_n\}, U_n, ...)
\]
is a play of the game \( P_1(X_Y) \) where ONE follows \( \sigma \) and let
\[
(W_0, V_0, ..., W_n, V_n, ...)
\]
be the corresponding play in the game \( BM(X) \). To see that \( \{q_i : i \in \omega\} \) is closed discrete, fix \( y \in Y \) and fix \( n \) such that \( y \in C_n \). But as \( \{q_i : i \geq n\} \subseteq V_n \) and since \( \overline{V_n} \cap C_n = \emptyset \), it follows that \( y \) is not an accumulation point of \( \{q_i : i \in \omega\} \). Now since \( \tau \) is winning for TWO in \( BM(X) \) we have that \( \cap_{n \in \omega} W_n \neq \emptyset \) and as \( W_n \subseteq U_n \) for each \( n \) it follows that \( \cap_{n \in \omega} U_n \neq \emptyset \) so the play \( (\{q_0\}, U_0, ..., \{q_n\}, U_n, ...) \) is won by player ONE.

**Corollary 14** Let \( X = RQ \) be the Michael line. Then ONE has a winning strategy in all of the games \( CM(X), P_1(X), CP(X) \) and \( P_0(X) \).

**Theorem 15** If \( X \) is a second countable regular space with no isolated points such that for every countable dense subset \( Y \subseteq X \) TWO does not have a winning strategy in \( CP(X_Y) \) (or any of the other games \( P_0(X_Y), CM(X_Y), \) or \( P_0(X_Y) \)) then \( X \) is a Baire space.

**Proof** : Again, it suffices to prove this for the games \( CP(X_Y) \) an \( P_0(X_Y) \). We will use that if \( X \) is second countable and \( Y \subseteq X \) is countable, then \( X_Y \) is normal. Let \( D_1 \supseteq D_2 \supseteq ... \supseteq D_n \supseteq ... \) be a sequence of dense open subsets of \( X \). For each \( n \) let \( S_n \) be a countable dense subset of \( D_n \). Find a single dense subset \( Y \) of \( X \) such that for each \( n \), \( Y \setminus S_n \) is finite, and \( Y \subseteq D_1 \).

We shall now play \( CM(X_Y) \). As in Theorem 10’s proof, we may assume that ONE’s moves are subsets of \( Y \) (i.e., if TWO does not have a winning strategy, then TWO does not have a winning strategy even if ONE plays subsets of \( Y \) only).

TWO now plays the game \( CM(X_Y) \) according to the following strategy: For each \( n \), put \( F_n = Y \setminus D_n \). Then each \( F_n \) is a finite set. TWO plays open sets whose closures are subsets of \( D_1 \) repeatedly, until ONE’s selected \( X_Y \)-closed subset of \( Y \) is disjoint from \( F_2 \). Then TWO computes the largest
such that $F_1 \cup \ldots \cup F_{m_1}$ is disjoint from ONE’s chosen set, at which point TWO’s move will be open sets whose closures are subsets of $D_{m_1}$. TWO continues playing with respect to this $D_{m_1}$ until ONE’s chosen set is disjoint from $F_1 \cup \ldots \cup F_{m_1+1}$. Then TWO computes the largest $m_2$ such that $F_1 \cup \ldots \cup F_{m_2}$ is disjoint from ONE’s most recent move. TWO then plays open sets whose closures are contained in $D_{m_2}$ until ONE’s chosen set is disjoint from $F_1 \cup \ldots \cup F_{m_2+1}$, and so on.

By hypothesis, this is not a winning strategy for TWO of $CM(X \setminus Y)$. Thus, there is a play according to this strategy which is won by ONE. For such a play we have $\cap_{n<\infty} D_n \setminus Y \neq \emptyset$.

The proof for $P_0(X \setminus Y)$ is similar.

**Corollary 16** Let $G$ denote any of the games $CP$, $CM$, $P_0$ or $P_1$. If $X$ is a second countable regular space with no isolated points such that for every nonempty open subset $U$ of $X$ and every countable dense subset $Y \subseteq U$ TWO does not have a winning strategy in $G(U \setminus Y)$, then $X$ is a Baire space (i.e., ONE does not have a winning strategy in $BM(X)$).

Now we present an example demonstrating that the games can be undetermined. The construction of this example depends the axiom ♦. The following formulation of ♦ is but one of many equivalent formulations (see [9] Exercise II.51). ♦ is the assertion that there is a sequence $(g_\alpha : \alpha < \omega_1)$ such that

1. $g_\alpha$ is a function from $<\omega \alpha$ to $\alpha$, and
2. for each function $f$ from $<\omega_1$ to $\omega_1$ the set $\{\alpha < \omega_1 : f|^{<\omega} \alpha = g_\alpha\}$ is stationary.

**Theorem 17** (♦) There is a Lusin set $L \supseteq Q$ such that ONE does not have a winning strategy in $CP(L \setminus Q)$.

**Proof**: From now on assume ♦. Fix bijective enumerations

1. $(H_\gamma : \gamma < \omega_1)$ of the dense open subsets of $R$;
2. $(V_\gamma : \gamma < \omega_1)$ of the nonempty open subsets of $R$; Assume that $\{V_n : n \in \omega\}$ enumerates all sets of the form $R \setminus B$ where $B$ is finite union of open intervals with rational endpoints;
3. \((C_\gamma : \gamma < \omega_1)\) of the closed, nonempty subsets of \(Q\).

By ♦ fix for each \(\beta < \omega_1\) a function \(F_\beta\) from \(<\omega \{V_\gamma : \gamma < \beta\}\) to \(\{C_\gamma : \gamma < \beta\}\) such that for each strategy \(F\) of ONE the set \(\beta < \omega_1 : F[<\omega \{V_\gamma : \gamma < \beta\} = F_\beta]\) is a stationary subset of \(\omega_1\).

We shall obtain \(L \supseteq Q\) by choosing distinct irrational numbers \(z_\delta, \omega \leq \delta < \omega_1\), so that \(L = Q \cup \{z_\delta : \omega \leq \delta < \omega_1\}\) will have the required properties. To do this we select by recursion for each \(\gamma < \omega_1\) a function \(F_\gamma\) from \(\langle \omega, \{V_\gamma : \gamma < \beta\} \rangle\) to \(\{C_\gamma : \gamma < \beta\}\) such that for each strategy \(F\) of ONE the set \(\beta < \omega_1 : F[\langle \omega, \{V_\gamma : \gamma < \beta\} \rangle = F_\beta]\) is a stationary subset of \(\omega_1\).

We shall obtain \(L \supseteq Q\) by choosing distinct irrational numbers \(z_\delta, \omega \leq \delta < \omega_1\), so that \(L = Q \cup \{z_\delta : \omega \leq \delta < \omega_1\}\) will have the required properties. To do this we select by recursion for each \(\gamma < \omega_1\) a function \(F_\gamma\) from \(\langle \omega, \{V_\gamma : \gamma < \beta\} \rangle\) to \(\{C_\gamma : \gamma < \beta\}\) such that for each strategy \(F\) of ONE the set \(\beta < \omega_1 : F[\langle \omega, \{V_\gamma : \gamma < \beta\} \rangle = F_\beta]\) is a stationary subset of \(\omega_1\).

We shall obtain \(L \supseteq Q\) by choosing distinct irrational numbers \(z_\delta, \omega \leq \delta < \omega_1\), so that \(L = Q \cup \{z_\delta : \omega \leq \delta < \omega_1\}\) will have the required properties. To do this we select by recursion for each \(\gamma < \omega_1\) a function \(F_\gamma\) from \(\langle \omega, \{V_\gamma : \gamma < \beta\} \rangle\) to \(\{C_\gamma : \gamma < \beta\}\) such that for each strategy \(F\) of ONE the set \(\beta < \omega_1 : F[\langle \omega, \{V_\gamma : \gamma < \beta\} \rangle = F_\beta]\) is a stationary subset of \(\omega_1\).

Before verifying that this recursion can be accomplished, we verify that a set \(L\) produced by it has the required properties. Thus, assume that \(L\) has been defined using this recursion. Since \((H_\gamma : \gamma < \omega_1)\) enumerates all the dense open sets of \(R\), clause 1 implies that \(L\) is a Lusin set. Next, let \(F\) be a strategy for ONE in \(\text{CM}(L_Q)\). We may assume that the values of \(F\) are closed subsets of \(Q\), and that if

\[F(\emptyset), U_1, F(\emptyset, U_1), U_2, F(\emptyset, U_1, U_2), U_3, \ldots\]

is any \(F\)-play, then

\[\cap_{n<\infty} F(\emptyset, U_1, \ldots, U_n) = \emptyset.\]

Furthermore we may assume that \(U_n\) are open subsets of \(R\). By a standard closing off argument, it is easy to show that the set \(C\) of \(\gamma\) for which \(\{V_\delta : \delta < \gamma\}\) is closed under finite intersections and \(\{R \setminus \{z_\delta\} : \delta < \gamma\} \subseteq \{V_\delta : \delta < \gamma\}\)
is closed and unbounded. By ♦ there is a $\gamma \in C$ such that $F[\prec \{V_\delta : \delta < \gamma\}] = F_\gamma$. Therefore

$$(O_\gamma^0, U_\gamma^0, O_\gamma^1, U_\gamma^1, \ldots)$$

is a play according to the strategy $F$. Therefore $\bigcap_{n \in \omega} O_\gamma^0 = \emptyset$ and by the inductive hypotheses we have

1. $\bigcup_{n \in \omega} U_\gamma^0 \cap (Q \cup \{z_\delta : \delta < \gamma\}) = \emptyset$,
2. $T_\gamma \cap \bigcup_{n \in \omega} U_\gamma^n = \emptyset$, and
3. $z_\delta \in T_\gamma$ for each $\delta \geq \gamma$.

Therefore $\bigcap_{n \in \omega} U_\gamma^n = \emptyset$ and TWO wins this play of the game.

Therefore it suffices to show how to carry out the recursive construction. So let $\gamma$ be given and consider the potential partial strategy $F_\gamma$. If either $\{V_\delta : \delta < \gamma\}$ is not closed under finite intersections or if $\{R \setminus \{z_\delta : \delta < \gamma\}\} \not\subseteq \{V_\delta : \delta < \gamma\}$ then let $T_\gamma = R$ and choose $z_\gamma \in \bigcap_{\delta \leq \gamma} (T_\delta \cap H_\delta) \setminus (Q \cup \{z_\delta : \omega \leq \delta < \gamma\})$. The inductive hypothesis are easily preserved.

Otherwise we define the required play of the game as follows: Fix enumerations $Q = \{q_n : n \in \omega\}$ and $\{z_\delta : \delta < \gamma\} = \{x_n : n \in \omega\}$. Let $O_0^\gamma = F_\gamma(\emptyset)$ and let $U_0^\gamma \in \{V_\delta : \delta < \gamma\}$ using our assumption so that

1. $x_0 \notin U_0^\gamma$ and
2. if $q_0 \notin O_0^\gamma$ then $q_0 \notin \overline{U_0^\gamma}$.

We can do this by our assumption on $\{V_\delta : \delta < \gamma\}$. Having defined $(O_i^\gamma, U_i^\gamma)_{i < n}$ let $O_n^\gamma = F_\gamma(U_0^\gamma, \ldots, U_{n-1}^\gamma)$ and choose $U_n^\gamma \in \{V_\delta : \delta < \gamma\}$ using our assumption so that

1. $U_n^\gamma \cap \{x_i : i \leq n\} = \emptyset$, and
2. if $i \leq n$ and $q_i \notin O_n^\gamma$, then $q_i \notin \overline{U_n^\gamma}$.

Clearly, $\bigcap_{n \in \omega} U_n^\gamma \cap Q = \emptyset$ and $\bigcap_{n \in \omega} U_n^\gamma \cap \{z_\delta : \delta < \gamma\} = \emptyset$. Let $T_\gamma = \bigcup_{n \in \omega} U_n^\gamma \setminus \overline{U_n^\gamma}$ (where $U_{n-1}^\gamma = R$). We leave to the reader the easy verification that $T_\gamma$ and $(O_n^\gamma, U_n^\gamma)_{n \in \omega}$ are as required thus completing the proof.

An obvious modification of the above construction yields a Lusin set $L$ on which ONE also does not have a winning strategy in the game $P_0(X_Y)$ (hence in any of the games).
Corollary 18 (◊) There is a set \( L \supset Q \) for which all of the games on \( L_Q \) are undetermined.

**Proof**: Let \( L \) be an appropriate Lusin set on which ONE does not have a winning strategy in the any of the games. Since a Lusin set is a Baire space, Theorem 11 implies that TWO also has no winning strategy in any of the games on \( L_Q \).

**Question 4** Are there sets of reals \( Y \subseteq X \) so that any of the games \( CM(X_Y) \), \( CP(X_Y) \), \( P_0(X_Y) \) or \( P_1(X_Y) \) is undetermined?

Also we have no example of a Michael type line for which any of the games are different. There are quite a few versions of the following question that are open:

**Question 5** Are there sets of reals \( Y \subseteq X \) for which, for example, ONE has a winning strategy in \( CM(X_Y) \) but ONE does not have a winning strategy in \( P_0(X_Y) \)?

We don’t know if ◊ is needed for Theorem 17:

**Question 6** Can there be a Lusin set \( L \supset Q \) such that ONE has a winning strategy in, for example, \( CM(L_Q) \)?

Notice that if \( L \) is a Lusin set, and \( Q \subseteq L \), then TWO has a strategy in the game \( CP(L_Q) \) such that if

\[
(O_0, U_0, O_1, U_1, ...) 
\]

is a play according to the strategy \( \sigma \), then \( \bigcap_{n \in \omega} U_n \) is countable.

**Morita P-spaces**

A space \( X \) is called a Morita P-space or simply a P-space if for each infinite \( \kappa \) and for each family

\[
\{ G(s) : s \in \kappa^{<\omega} \}
\]

of open subsets of \( X \) such that \( G(s|n) \subseteq G(s) \) whenever \( s \in \kappa^{<\omega} \) and \( n \in \text{dom}(s) \), there exists a family

\[
\{ F(s) : s \in \kappa^{<\omega} \}
\]

of closed subsets of \( X \) satisfying the following conditions
1. \( F(s) \subseteq G(s) \) for each \( s \in \kappa^{<\omega} \).

2. For each \( f \in \kappa^{\omega} \), if \( \bigcup \{ G(f|n) : n \in \omega \} = X \) then \( \bigcup \{ G(f|n) : n \in \omega \} = X \).

The game \( CM(X) \) was introduced by R. Telgárszky in [19] where he proved the following theorem.

**Theorem 19** TWO has a winning strategy in \( CM(X) \) if and only if \( X \) is a Morita P-space.

Morita proved \( X \) is a normal P-space if and only if for every \( \kappa \) and every \( M \subseteq \kappa^{\omega} \), \( X \times M \) is normal (equivalently \( X \times M \) is normal for every metric space \( M \)) [10]. In particular, if TWO does not have a winning strategy in \( CM(X) \) then \( X \times M \) is not normal for some subspace \( M \) of \( \kappa^{\omega} \). Examining the proof of this last consequence of Morita’s theorem gives the following theorem.

**Theorem 20** If ONE has a winning strategy in \( CM(X) \) then there is an infinite \( \kappa \) such that \( X \times \kappa^{\omega} \) is not normal.

**PROOF:** Let \( \sigma \) be a winning strategy for ONE in \( CM(X) \). Let \( \kappa \) be minimal such that for each closed subset \( F \) of \( X \), \( \chi(F,X) \leq \kappa \) and let \( (U^F_\alpha : \alpha < \kappa) \) be an open neighborhood base at \( F \) in \( X \). We now define a family of closed sets \( (F_s : s \in \kappa^{<\omega}) \) by recursion. Let \( F_\emptyset = \sigma(\emptyset) \). If \( s \in \kappa^n \) and if \( F_{s|k} \) has been defined for each \( k < n \), let

\[
F_s = \sigma(F_\emptyset, U^{F_\emptyset}_{s(0)}, F_{s|1}, U^{F_{s|1}}_{s(1)}, \ldots, U^{F_{s|n-1}}_{s(n)}).
\]

Notice that the family defined has the property that if \( f \in \kappa^{\omega} \) then

\[
(F_\emptyset, U^{F_\emptyset}_{f(0)}, F_f_{f(1)}, U^{F_{f(1)}}_{f(1)}, \ldots, U^{F_{s(n)}}_{s(n)}), F_f, \ldots)
\]

is a play of the game \( CM(X) \) where ONE uses the strategy \( \sigma \). As \( \sigma \) is a winning strategy for ONE, we have

\[
\bigcap_{n \in \omega} F_{f|n} = \emptyset
\]

for each \( f \in \kappa^{\omega} \).

The following lemma is a special case of Theorem 4.1 in [13].
Lemma 21 If $X \times \kappa^\omega$ is normal, then there is a family of open sets $(U_s : s \in \kappa^{<\omega})$ such that $F_s \subseteq U_s$ for every $s \in \kappa^{<\omega}$ and $\bigcap_{n \in \omega} U_{f|n} = \emptyset$ for every $f \in \kappa^\omega$.

Therefore, to finish the proof of the theorem, it suffices to show that there is no such family of open sets. So suppose that $(U_s : s \in \kappa^{<\omega})$ is such that $F_s \subseteq U_s$ for every $s \in \kappa^{<\omega}$. For each $s \in \kappa^{<\omega}$ fix $\alpha(s) \in \kappa$ so that $U_{\alpha(s)}^F \subseteq U_s$. By recursion define $f \in \kappa^\omega$ such that $f(n) = \alpha(f|n)$ for each $n \in \omega$. Since the strategy $\sigma$ is winning, ONE wins the play of the game

$$(F_\emptyset, U_{\emptyset}^F, F_{f(0)}^F, U_{f(1)}^F, \ldots, U_{f(n)}^F, \ldots).$$

And since $\bigcap F_{f|n} = \emptyset$ we have that $\bigcap_{n \in \omega} U_{f|n}^F \neq \emptyset$. Therefore $\bigcap_{n \in \omega} U_{f|n} \neq \emptyset$ as required.

Consistently, the converse to Theorem 20 does not hold. Indeed, if $L$ is the Luzin set constructed from $\Diamond$ in the previous section, then by Corollary 18 ONE does not have a winning strategy in any of the games. However, $L \mathcal{Q} \times M$ is not normal for any $M \subseteq R \setminus \mathcal{Q}$ containing $L \setminus \mathcal{Q}$. In particular $L \mathcal{Q} \times \omega^\omega$ is not normal.

Perfect spaces. A space is said to be perfect if every closed subset is a $G_\delta$. It is well known that every perfect space is countably metacompact and that every perfectly normal space is a Morita P-space [10]. Indeed, every perfect space is a P-space. For completeness sake we give a proof.

Theorem 22 Suppose that $X$ is perfect, then TWO has a winning strategy in $CM(X)$.

Proof: For each closed subset $C \subset X$ fix open sets $V_n(C)$ for each $n \in \omega$ such that $\bigcap_{n \in \omega} V_n(C) = C$. Define a strategy $\sigma$ for player TWO in the game $CM(X)$ as follows. Suppose that $(C_0, U_0, \ldots, C_n, U_n)$ is a partial play of the game. Let $\sigma(C_0, U_0, \ldots, C_n, U_n) = \bigcap_{k \leq n} V_n(C_k)$. Suppose that $(C_0, U_0, \ldots, C_n, U_n, \ldots)$ is a play of the game where TWO follows the strategy $\sigma$. Suppose $\bigcap_n C_n = \emptyset$ (otherwise TWO wins and we are done). Then, for each $x$ fix $n$ such that $x \not\in C_n$. Also fix $m \geq n$ such that $x \not\in V_m(C_n)$. Then by definition $x \not\in U_m$. Therefore TWO wins the play of the game.

Corollary 23 TWO has a winning strategy in $CP(X)$ for any perfectly normal space $X$. 

**Proof:** If $X$ is normal, then by Theorem 4, TWO has a winning strategy in $CP(X)$ if and only if TWO has a winning strategy in $CM(X)$.

The connection between perfect spaces and the partition games is less clear.

**Question 7** If $X$ is perfect, does TWO have a winning strategy in $P_1(X)$? If $X$ is perfectly normal, does TWO have a winning strategy in $P_0(X)$?

The converse of Theorem 22 is not true: $\omega_1$ is not perfect but since it is countably compact, TWO has trivial winning strategies in all of the games.

For certain classes of spaces $X$ and depending on your set theory, points being $G_\delta$ in $X$ may imply that closed discrete subsets of $X$ are $G_\delta$-sets. For example, in many models of set theory, first countable, countably paracompact spaces are collectionwise Hausdorff (see [17] for a survey). Certainly if a closed discrete set is separated in a first countable space, then it is a $G_\delta$-set. The weaker assumption of countable metacompactness does not in general imply collectionwise Hausdorff, but assuming PMEA, first countable, countably metacompact spaces do have the property that all closed discrete subsets are $G_\delta$-sets [2]. It is an open question whether the large cardinal inherent in the assumption of PMEA (PMEA implies that the continuum is real valued measurable) is necessary to establish the consistency of this theorem. Therefore it is natural to consider whether strengthening the assumptions on the space give theorems in ZFC or in the absence of large cardinals. For example we have the following collection of questions.

**Question 8** Suppose that $X$ is a first countable space and that TWO has a winning strategy in one or more of the four games $P_0(X)$, $P_1(X)$, $CP(X)$ and $CM(X)$. Are closed discrete subsets of $X$ $G_\delta$-sets?

One version of this question was asked by Nyikos in [11]. There he asked whether first countable (normal) Morita P-spaces have the property that closed discrete subsets are $G_\delta$-sets.

Suppose that $X$ is first countable and that $D \subseteq X$ is closed discrete. Let $X_D$ denote the space obtained by isolating all the points of $X \setminus D$. Then $D$ is a $G_\delta$-set in $X$ if and only if $X_D$ is perfect. If $S$ denotes any of the four games, then it is easy to see that TWO has a winning strategy in $S(X)$ implies that TWO has a winning strategy in $S(X_D)$. Therefore we may restrict our attention to spaces of scattered height two.
The following example was considered by D.K. Burke as an example of a countably metacompact $T_2$ space with a closed discrete non $G_\delta$-set [3]. A $Q$-set is an uncountable subset of reals for which every subset is a relative $G_\delta$. It is consistent with ZFC that there are $Q$-sets (MA+¬CH implies every subset of reals of size less than the continuum is a $Q$-set) and it is consistent with ZFC that there are no $Q$-sets (for example under CH).

**Example 24** Suppose that there is a $Q$-set. Then there is a first countable $T_2$ space $X$ of scattered height two such that TWO has a winning strategy in $\text{CM}(X)$ but $X$ is not countably paracompact (hence neither perfect nor normal).

**Proof:** Let $Y \subset \mathbb{R}$ be a $Q$-set. Let $X = Y \times \{0,1\}$ be two copies of $Y$. We topologize $X$ so that for each $y \in Y$

1. $\{(y,0)\}$ is open, and

2. for every open interval $J \subset \mathbb{R}$ such that $y \in J$,

$$\{(y,1)\} \cup ((Y \cap J \setminus \{y\}) \times \{0\})$$

is a basic open set containing $(y,1)$.

$X$ is $T_2$, and $X_1 = Y \times \{1\}$ is a closed discrete subset of $X$ that is not a $G_\delta$-set. Also $X$ is countably metacompact but not countably paracompact (see [3]). We can prove a bit more:

**Lemma 25** TWO has a winning strategy in the game $\text{CM}(X)$.

**Proof:** Since any subset of $X_0 = Y \times \{0\}$ is open, we may assume ONE will only play subsets of $X_1$. Since $Y$ is a $Q$-set, for each $D \subset Y$ there are open subsets $U_k(D)$ of $Y$ such that $\bigcap_{k \in \omega} U_k(D) = D$. Let us describe a strategy $\sigma$ for TWO: Let $C_0 \supset \ldots \supset C_n$ be a sequence of subsets of $X_1$. Observe that $C_i = D_i \times \{1\}$ for appropriate subsets $D_i \subset Y$. Let

$$\sigma(C_0, \ldots, C_n) = C_n \cup (\bigcap_{i \leq n} U_n(D_i) \times \{0\})$$

Then $\sigma$ is a winning strategy for TWO.

The previous example is not regular. However, assuming in addition to the $Q$-set that $b = \omega_1$ then there is a zero-dimensional modification of this
example exhibiting all the same properties (see [15]). We don’t know whether TWO has a winning strategy in $P_1(X)$ (for either the $T_2$ version of the space or its zero-dimensional modification). However as both examples fail to be countably paracompact, ONE has a winning strategy in both games $\text{CP}(X)$ and $P_0(X)$.

There essentially only three other examples of countably metacompact spaces containing closed discrete sets that are not $G_\delta$-sets. Two are spaces of scattered height two constructed via forcing. The first of these examples is due to Shelah (see [14]). This space is normal hence by Theorem 5, TWO has a winning strategy in the game $P_1$. However we do not know whether TWO has a winning strategy in any of the other games on this space. The second example is due to Z. Balogh and D.K. Burke [1]. It is not normal. It also fails to be countably paracompact so ONE has a winning strategy in the games $\text{CP}$ and $P_0$. However it is a Morita P-space [1] so TWO has a winning strategy in $CM$.

The third example was constructed from a strong version of $\diamondsuit$ [16]. We can show that for a minor modification of this space, TWO does not have a winning strategy in $P_1$. We conjecture that TWO does not have winning strategies in either game $P_1$ or $CM$. Considering all these examples, the most interesting open questions are

**Question 9** Suppose that $X$ is a first countable Tychonoff space.

1. If TWO has a winning strategy in $\text{CP}(X)$ or $P_0(X)$ is each closed discrete subset of $X$ a $G_\delta$-set?

2. Assume $V = L$. If TWO has a winning strategy in $\text{CM}(X)$ or $P_1(X)$ is each closed discrete subset of $X$ a $G_\delta$-set?

3. If TWO has a winning strategy in both games $\text{CM}(X)$ and $P_1(X)$ is each closed discrete subset of $X$ a $G_\delta$-set?

**Markov Strategies and Generalized Metric Spaces.** In a two player game $G$ it is natural to consider strategies that do not require full memory. A strategy for either player that depends only on the last move by the other player and the inning of the game is called a *Markov strategy*. It turns out that spaces for which TWO has a winning Markov strategy in the games thus far considered coincide with well known classes of generalized metric
spaces. See either [7] or [8] for more on the spaces we will be considering. Furthermore, we show that the existence of winning Markov strategies for TWO does imply that closed discrete sets are $G_\delta$-sets.

The analogue to Theorem 1 for Markov strategies also holds and the strategies described in Theorem 2 are all Markov strategies.

**Definition 26** $(X, \tau)$ is called a $\beta$-space if there is a function $g : \omega \times X \to \tau$ such that

1. $x \in g(n, x)$ for each $x \in X$.

2. If $x \in g(n, x_n)$ for each $n \in \omega$, then $\{x_n, n \in \omega\}$ has a cluster point in $X$.

Many important classes of spaces are $\beta$-spaces. For example stratifiable, wN-spaces and monotonically normal spaces are $\beta$-spaces. As well, all Moore spaces are $\beta$-spaces and all $\beta$-spaces are Morita P-spaces hence are countably metacompact. This last fact is also a consequence of the following theorem characterizing $\beta$-spaces in terms of the countably metacompact partition game.

**Theorem 27** Suppose that $X$ is a $T_1$ space. Then $X$ is a $\beta$-space if and only if TWO has a winning Markov strategy in $P_1(X)$. Furthermore if $X$ is a $\beta$-space, then TWO has a winning Markov strategy in $CM(X)$.

**Proof:** Given that $X$ is a $\beta$ space, a Markov strategy for TWO in both games is given by $\sigma(C, n) = \bigcup_{x \in C} g(n, x)$. To see this, suppose $(C_n)_{n \in \omega}$ is ONE’s sequence of plays in either game such that TWO’s sequence of plays $(\sigma(C_n, n))_{n \in \omega}$ is not point finite. Then there is a sequence of points $x_n \in C_n$ and a point $x \in X$ such that $x \in g(n, x_n)$ for each $n \in \omega$. Therefore $\{x_n : n \in \omega\}$ has a cluster point, and this implies, in the case of the game $CM(X)$, that $\bigcap_n C_n \neq \emptyset$; and, in the case of the game $P_1(X)$, that $\{C_n : n \in \omega\}$ is not discrete.

For the converse let $\sigma$ be a winning Markov strategy for TWO in $P_1(X)$. For each $n \in \omega$ and each $x \in X$, let $g(n, x) = \sigma(\{x\}, n)$. To see that $g$ witnesses that $X$ is a $\beta$-space, fix $\{x_n : n \in \omega\} \subset X$ and $x \in X$ such that $x \in g(n, x_n)$ for each $n \in \omega$. Since $\sigma$ is winning for TWO, $\{x_n : n \in \omega\}$ is not discrete. Therefore it has a cluster point.

Therefore, for Markov strategies we get a partial answer to Question 3.
Corollary 28 If TWO has a winning Markov strategy in $P_1(X)$ then TWO has a winning Markov strategy in $CM(X)$.

And as a corollary to the proof of Theorem 27 we have

Corollary 29 TWO has a winning Markov strategy in $P_1(X)$ if and only if TWO has a winning Markov strategy in the version of $P_1(X)$ where ONE chooses singleton sets.

Definition 30 A space $(X, \tau)$ is said to be wN (=weak Nagata) if there is a function $g: \omega \times X \to \tau$ such that

1. $x \in g(n, x)$ for each $n \in \omega$.

2. If there is an $x \in X$ such that $g(n, x) \cap g(n, x_n) \neq \emptyset$ for each $n \in \omega$ then $(x_n : n \in \omega)$ has a cluster point.

The following is not the usual definition of stratifiable but is equivalent (see 5.8 in [7]).

Definition 31 A space $(X, \tau)$ is said to be stratifiable if there is a function $g: \omega \times X \to \tau$ such that

1. $\{x\} = \bigcap_n g(n, x)$.

2. If $x \in g(n, x_n)$ for all $n \in \omega$, then $(x_n) \to y$.

3. For each closed set $H$ and each $y \not\in H$, there is an $n \in \omega$ such that $y \not\in \bigcup\{g(n, x) : x \in H\}$.

Theorem 32 If $X$ is either a wN-space or a stratifiable space then TWO has a winning Markov strategy in the games $CP(X)$ and $P_0(X)$.

Proof: Suppose $g$ is a function witnessing either that $X$ is wN or stratifiable. A winning Markov strategy for TWO in both games is given by $\sigma(K, n) = \bigcup_{x \in K} g(n, x)$.

While we know by Theorems 5 and 6 that it may be the case that TWO has a winning strategy in $P_1(X)$ while ONE has a winning strategy in $CM(X)$, the existence of Markov strategies seems less pathological. However we don’t know the answers to the following questions:
Question 10 1. Are there any implications between ‘TWO has a winning Markov strategy in CP(X)’ and ‘TWO has a winning Markov strategy in P0(X)?’

2. If TWO has a winning Markov strategy in CM(X) does TWO also have a winning Markov strategy in P1(X)?

It would also be interesting to answer these questions for a restricted class of spaces (e.g., for normal spaces).

It is easy to show that if \( g \) is the function witnessing that \( X \) is a \( \beta \)-space and if, in addition, \( \cap_n g(n, x) = \{ x \} \) for each \( x \in X \) (i.e., points are \( G_\delta \)), then closed discrete subsets of \( X \) are \( G_\delta \). So we get in ZFC the following partial answer to Question 9.

**Theorem 33** Suppose that points are \( G_\delta \) in \( X \) and suppose that TWO has a winning Markov strategy in \( P_1(X) \). Then closed discrete subsets of \( X \) are \( G_\delta \)-sets.

However, we don’t know the answer to the following:

**Question 11** If \( X \) is first countable and TWO has a winning Markov strategy in CM(X), is each closed discrete subsets of \( X \) a \( G_\delta \)-set?

**References**


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