Open covers and square bracket partition relations.

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Abstract

Let $X$ be an infinite separable metric space. An open cover $\mathcal{U}$ of $X$ is an $\omega$-cover if $X \not\in \mathcal{U}$ and for every finite subset $F$ of $X$ there is a $U \in \mathcal{U}$ such that $F \subseteq U$. Let $\Omega$ be the collection of $\omega$-covers of $X$.

We show that the partition relation $\Omega \rightarrow [\Omega]^2_2$ holds if, and only if, the partition relation $\Omega \rightarrow [\Omega]^2_3$ holds.

For a set $S$ and for a positive integer $n$ the symbol $[S]^n$ denotes the collection of $n$-element subsets of $S$. Let $\mathcal{A}$ and $\mathcal{B}$ be two classes of objects and let $\ell, k$ and $n$ be positive integers with $\ell < k$. Then the symbol

$$\mathcal{A} \rightarrow [\mathcal{B}]^n_k/\leq$$

denotes the statement:

for every element $A$ of $\mathcal{A}$ and for every function

$$f : [A]^n \rightarrow \{0, 1, \ldots, k - 1\}$$

there is a subset $B$ of $A$ and an $J \in [[0, 1, \ldots, k - 1]]^{\leq \ell}$ such that $B$ is an element of $\mathcal{B}$, and $f(X) \in J$ whenever $X$ is an $n$-element subset of $B$.

The negation of this statement is denoted $\mathcal{A} \not\rightarrow [\mathcal{B}]^n_k/\leq$. In the case when $\ell$ is $k - 1$, we write $\mathcal{A} \rightarrow [\mathcal{B}]^n_k$ to denote $\mathcal{A} \rightarrow [\mathcal{B}]^n_{k/\leq}$.

These symbols have been introduced in [5] in the context of cardinality. It denotes a binary relation between elements of $\mathcal{A}$ and of $\mathcal{B}$, and is known as “the square bracket partition relation”. A special case of Ramsey’s famous theorem [7] can be stated in this notation as

$$\aleph_0 \rightarrow [\aleph_0]^2_2.$$

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These symbols have been used in several other contexts too: An unpublished theorem of Galvin states $\eta \rightarrow [\eta]_3^2$ — this means that for every coloring of two-elements sets of rational numbers, there is a subset of the set of rational numbers which has the same order type as the set of rational numbers, and one of the three colors, such that none of the two-element subsets of the given set has that color. These symbols have been extensively studied in connection with ultrafilters. Though these have been studied also on uncountable sets, I shall restrict my comments here to ultrafilters on the set of positive integers. A nonprincipal ultrafilter $\mathcal{U}$ is said to be a Ramsey ultrafilter if it satisfies $\mathcal{U} \rightarrow [\mathcal{U}]_3^2$. These were discussed for example in [3].

In [2] Blass introduces the notion of a weakly Ramsey ultrafilter: a nonprincipal ultrafilter $\mathcal{U}$ on the set of positive integers is said to be weakly Ramsey if it satisfies the partition relation $\mathcal{U} \rightarrow [\mathcal{U}]_3^2$. He then showed that the Continuum Hypothesis implies that there is a weakly Ramsey ultrafilter which is not a Ramsey ultrafilter. In other words, the implication

$$\mathcal{U} \rightarrow [\mathcal{U}]_3^2 \Rightarrow \mathcal{U} \rightarrow [\mathcal{U}]_3^2$$

is not provably reversible. Along these lines D. Devlin (using for example the Continuum Hypothesis) gave examples in [4] of nonprincipal ultrafilters $\mathcal{U}$ for which the relation $\mathcal{U} \rightarrow [\mathcal{U}]_6^3$ holds, while $\mathcal{U} \rightarrow [\mathcal{U}]_5^3$ fails. Analogues of this situation also occur for cardinal numbers ... it is known that when $\kappa$ is a singular cardinal number, then for all integers $n \geq 2$

$\kappa \not\rightarrow [\kappa]_{2^n-1}^n$ (see [11], Theorem 2.7.8), while when $\kappa$ has countable cofinality, it can happen that $\kappa \rightarrow [\kappa]_{2^{\omega-1}+1}^n$ (see [11], Theorem 2.7.7).

The moral of the story seems to be that it often happens that $\mathcal{A} \not\rightarrow [\mathcal{B}]_k^2$, while $\mathcal{A} \rightarrow [\mathcal{B}]_{k+1}^2$. In [6], [9] and [10] partition relations for families of open covers of topological spaces were introduced, and it was shown that these characterize some of the classical small sets of real numbers introduced by Rothberger, Menger and Hurewicz. It was also shown that there is a strong analogy between the combinatorial properties of collections of open covers for these sorts of spaces, and combinatorial properties of classical sorts of ultrafilters on the set of positive integers.

For example: Let $X$ be an infinite separable metric space. Then $X$ is said to have the Rothberger property if there is for every sequence $(\mathcal{U}_n : n < \omega)$ of open covers of $X$ a sequence $(\mathcal{U}_n : n < \omega)$ such that for each $n \mathcal{U}_n \in \mathcal{U}_n$, and...
\( \{ U_n : n < \omega \} \) is an open cover of \( X \). This property was introduced by Rothberger in [8]. An open cover \( U \) of \( X \) is said to be an \( \omega \)-cover if \( X \) is not a member of \( U \), but for each finite subset \( F \) of \( X \) there is an element \( U \) of \( U \) such that \( F \subseteq U \). We shall use the symbol \( \Omega \) to denote the collection of \( \omega \)-covers of \( X \). Results of [6] and [9] show that the partition relation \( \Omega \rightarrow [\Omega]_3^2 \) holds if, and only if, all finite powers of \( X \) have Rothberger’s property.

In light of what occurs in analogous situations, it is natural to now ask if one could have spaces for which \( \Omega \rightarrow [\Omega]_3^2 \) holds while \( \Omega \rightarrow [\Omega]_2^2 \) fails. The purpose of this note is to show that each of these partition relations implies the other. The result gives, via the methods of [1], a few more characterizations of those sets whose finite powers have the Rothberger property.

**Theorem 1** Let \( X \) be an infinite separable metric space. If \( \Omega \rightarrow [\Omega]_3^2 \), then for every positive integer \( k \), \( \Omega \rightarrow [\Omega]_{k/2}^2 \).

**Proof**: We induct on \( k \). The theorem is true for \( k \leq 3 \). Assume that \( k \geq 3 \) and that the implication has been verified up to \( k \). Let \( U \) be an \( \omega \)-cover of \( X \) and let \( f : [U]^2 \rightarrow \{0, 1, \ldots, k\} \) be given. Define \( g : [U]^2 \rightarrow \{0, 1, 2\} \) so that

\[
g([U, V]) = \begin{cases} 
0 & \text{if } f([U, V]) = 0 \\
1 & \text{if } f([U, V]) = 1 \\
2 & \text{if } f([U, V]) > 1.
\end{cases}
\]

Apply \( \Omega \rightarrow [\Omega]_3^2 \) to find an \( \omega \)-cover \( V \subset U \) on which \( g \) is \( \leq 2 \)-valued. Then on \( V \) \( f \) is \( \leq k \)-valued, and we may apply the induction hypothesis to find an \( \omega \)-cover \( W \subset V \) such that on it \( f \) is \( \leq 2 \)-valued. \( \square \)

By the results of [6] and in [9] an infinite separable metric space’s family of \( \omega \)-covers satisfies the partition relation \( \Omega \rightarrow [\Omega]_2^2 \) if, and only if, it has the following Rothberger–like covering property:

For every sequence \( (U_n : n < \omega) \) of \( \omega \)-covers of \( X \) there is a sequence \( (U_n : n < \omega) \) such that for each \( n \) \( U_n \in U_n \), and such that \( \{U_n : n < \omega\} \) is an \( \omega \)-cover of \( X \).

We now prove
Theorem 2  If $X$ is an infinite separable metric space, then the following statements are equivalent:

1. $\Omega \to [\Omega]^2_2$.
2. $\Omega \to [\Omega]^2_3$.

Proof: We must show that 2 implies 1. Thus, let $(\mathcal{U}_n : n < \omega)$ be a sequence of $\omega$-covers of $X$. For each $n$, enumerate $\mathcal{U}_n$ bijectively as $(U^m_n : m < \omega)$.

Define a new $\omega$–cover $\mathcal{V}$ by

$$\{U^0_k \cap U^n_k \cap U^n_\ell : k, n, \ell < \omega\} \setminus \{\emptyset\}.$$ 

For each element $V$ of $\mathcal{V}$ choose a representation

$$V = U^0_k \cap U^n_k \cap U^n_\ell.$$ 

Define a partition $f : [\mathcal{V}]^2 \to \{0, 1, 2, 3, 4\}$ so that

$$f(\{U^0_k \cap U^{k_1}_n \cap U^{n_1}_\ell, \{U^0_{k_2} \cap U^{k_2}_{n_2} \cap U^{n_2}_\ell\}) = \begin{cases} 
0 & \text{if } k_1 = k_2 \text{ and } n_1 = n_2 \\
1 & \text{if } k_1 = k_2 \text{ and } n_1 < n_2 \\
2 & \text{if } k_1 < k_2 \text{ and } n_1 < n_2 \\
3 & \text{if } k_1 < k_2 \text{ and } n_1 > n_2 \\
4 & \text{if } k_1 < k_2 \text{ and } n_1 = n_2 
\end{cases}$$

By Theorem 1 and by 2, $\Omega \to [\Omega]^2_3/\leq_2$ holds. Choose an $\omega$-cover $\mathcal{W} \subset \mathcal{V}$ on which $f$ is two-valued. List $\mathcal{W}$ as

$$(U^0_{k_1} \cap U^{k_1}_{n_1} \cap U^{n_1}_\ell, U^0_{k_2} \cap U^{k_2}_{n_2} \cap U^{n_2}_\ell, \ldots)$$

according to the lexicographic order of the triples $(k_i, n_i, \ell_i)$ which occur in representations of elements of $\mathcal{W}$.

There are four main cases to be considered.

Case 1: $\{0, 1\} \cap f[[\mathcal{W}]^2] = \emptyset$.
In this case we have $k_1 < k_2 < \ldots < k_n < \ldots$, and so if we set $V_{k_i} = U^{k_i}_{n_i}$, and for $m \notin \{k_i : i = 1, 2, 3, \ldots\}$ we choose $V_m \in \mathcal{U}_m$ arbitrary, then the sequence $(V_m : m = 1, 2, 3, \ldots)$ constitutes an $\omega$-cover of $X$ and for each $m$ we have $V_m \in \mathcal{U}_m$.

Case 2: $\{0, 1\} \cap f[[\mathcal{W}]^2] = \{1\}$. 

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Subcase 2.1: $f([W]^2) \subseteq \{1, 2\}$.
In this case we see that $n_i \neq n_j$ whenever $i \neq j$. But then $W$ refines the sequence $(U_{k_i}^n : i = 1, 2, 3, \ldots)$, whence the latter constitutes an $\omega$-cover of $X$. This sequence can then be augmented to one of the form $(U_n : n < \omega)$ where for each $n$, $U_n \in \mathcal{U}_n$.

Subcase 2.2: $f([W]^2) \subseteq \{1, 3\}$.
We argue that this case also doesn’t occur. To see this, first observe that 3 cannot be attained only finitely many times, since then $W$ would be a refinement of a finite subset of $\mathcal{U}_0$, and thus not an $\omega$-cover. But it also cannot be attained infinitely many times. To see this, list the subscripts of the $U_k$’s occurring in the representations of elements of $W$ monotonically, say

\[ k_1 \leq k_2 \leq \ldots \leq k_j \leq \ldots \]

If 3 occurs infinitely often, then it happens infinitely often that $k_j < k_{j+1}$. Define $i_0 = 1$ and $i_{n+1} > i_n$ to be minimal such that $k_{i_{n+1}} > k_{i_n}$. Look at the subset

\[ \{U_{k_{ij}}^0 \cap U_{k_{n_{ij}}}^{k_{ij}} \cap U_{k_{l_{ij}}}^{m_{ij}} : j = 1, 2, 3, \ldots\} \]

of $W$. It is homogeneous of color 3, and thus we find

\[ n_{i_1} > n_{i_2} > \ldots > n_{i_j} > \ldots, \]

an infinite descending sequence of ordinals, a contradiction.

Subcase 2.3: $f([W]^2) \subseteq \{1, 3\}$.
If the value 4 is taken only finitely many times, then $W$ would be a refinement of a finite subset of $\mathcal{U}_0$, and thus not an $\omega$-cover of $X$. If the value 1 is taken only finitely many times, then all but finitely many of the terms $U_{k_i}^n$ could be assigned to distinct ones of the covers $\mathcal{U}_n$, and would constitute an $\omega$-cover, in which case we would be done. Thus we must treat the case where each of 1 and 4 is attained infinitely often. Then

\[ (k_1, n_1), (k_2, n_2), \ldots, (k_r, n_r), \ldots \]

forms (in the lexicographic order) a strictly increasing sequence such that for each $i$, either $k_i < k_{i+1}$ and $n_i = n_{i+1}$, or else $k_i = k_{i+1}$ and $n_i < n_{i+1}$.

Consider the two sequences

\[ (U_{n_i}^k : i = 1, 2, 3, \ldots \text{ and } k_{i-1} < k_i \text{ or } k_i < k_{i+1}) \]
which consists of certain middle terms of the three-set intersections composing the elements of $W$, and

$$(U_{n_i}^m : i = 1, 2, 3, \ldots \text{ and } n_{i-1} < n_i \text{ or } n_i < n_{i+1})$$

which consists of certain right-hand terms of the three-set intersections composing the elements of $W$. Since $W$ refines the totality of sets belonging to these two sequences, these two sequences constitute an $\omega$–cover of $X$.

But this $\omega$–cover is such that it could be partitioned into two–element sets each of which could be assigned to distinct terms of the original sequence $(U_n : n < \omega)$. Being an $\omega$–cover, we can find a new $\omega$–cover by selecting one term per two–element set. In this way we find a selector for the original sequence of $\omega$–covers in such a way that the selector is also an $\omega$–cover.

**Case 3:** $\{0, 1\} \cap f[[W]^2] = \{0\}$.

**Subcase 3.1:** $f[[W]^2] \subseteq \{0, 2\}$.

If the value 2 occurs only a finite number of times, then $W$ is a refinement of a finite subset of $U_0$, and thus not an $\omega$–cover of $X$. Thus, the value 2 is achieved infinitely often. Each time it is achieved and only then, both $k_i$ and $n_i$ increase in value. Let $i_1 < i_2 < \ldots$ be such that

- $i_1 = 1$, 
- for each $j$, if $i_j \leq t < i_{j+1}$, then $k_{i_j} = k_t$ and $n_{i_j} = n_t$, 
- for each $j$, $k_{i_j} < k_{i_{j+1}}$, and
- $\{(k_{i_j}, n_{i_j}) : j = 1, 2, 3, \ldots\} = \{(k_j, n_j) : j = 1, 2, 3, \ldots\}$.

For each $j$ put $V_{k_{i_j}} = U_{n_{i_j}}^{k_{i_j}}$ and for $n$ not in $\{k_{i_j} : j = 1, 2, 3, \ldots\}$, choose $V_n$ from $U_n$ arbitrarily. Then for each $n V_n \in U_n$, and $W$ is a refinement of $\{V_n : n = 1, 2, 3, \ldots\}$; thus the latter is an $\omega$–cover of $X$.

**Subcase 3.2:** $f[[W]^2] \subseteq \{0, 3\}$.

The value 3 can be attained only a finite number of times, lest we have an infinite descending sequence of ordinals. But then $W$ is a refinement of a finite subset of $U_0$, and so not an $\omega$–cover of $X$. It follows that this case doesn’t occur.

**Subcase 3.3:** $f[[W]^2] \subseteq \{0, 4\}$.

The value 4 must be attained an infinite number of times, else $W$ would be a refinement of a finite subset of $U_0$, hence not an $\omega$–cover of $X$. But then
an argument as in Subcase 3.1 shows that there is a sequence \((U_n : n < \omega)\) such that for each \(n \in U_n\), and \(\{U_n : n < \omega\}\) is an \(\omega\)-cover of \(X\).

**Case 4:** \(f([W]^{2}) \subseteq \{0, 1\}\). Then there is a fixed \(k\) such that each element of \(W\) is a subset of \(U_k\). Since \(X \neq U_k\) it follows that \(W\) doesn’t even cover \(X\). Consequently, this case doesn’t occur. \(\Box\)

**References**


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