Rothberger’s property and partition relations∗

Marion Scheepers

Let $X$ be an infinite but separable metric space. An open cover $U$ of $X$ is said to be large if, for each $x \in X$ the set $\{U \in U : x \in U\}$ is infinite. The symbol $\Lambda$ denotes the collection of large open covers of $X$. An open cover $U$ of $X$ is said to be an $\omega$–cover if for each finite subset $F$ of $X$ there is a $U \in U$ such that $F \subseteq U$, and $X$ is not a member of $U$. $X$ is said to have Rothberger’s property if there is for every sequence $(U_n : n = 1, 2, 3, \ldots)$ of open covers of $X$ a sequence $(U_n : n = 1, 2, 3, \ldots)$ such that:

1. for each $n$, $U_n$ is a member of $U_n$, and
2. $\{U_n : n = 1, 2, 3, \ldots\}$ is a cover of $X$.

Rothberger introduced this property in his paper [2]. For convenience we let $O$ denote the collection of all open covers of $X$.

In [3] it was shown that $X$ has Rothberger’s property if, and only if, the following partition relation is true for large open covers of $X$:

$$\Lambda \rightarrow_{\Lambda} (\Lambda, \text{not point-finite})^2.$$  \hspace{1cm} (1)

This partition relation means:

for every large cover $U$ of $X$, for every coloring

$$f : [U]^2 \rightarrow \{0, 1\}$$

such that for each $U \in U$ and each large cover $V \subset U$ there is an $i$ with $\{V \in V : f(\{U, V\}) = i\} \text{ a large cover of } X$,

either there is a large cover $W \subset U$ such that $f(\{A, B\}) = 0$ whenever $\{A, B\} \in [W]^2$,

or else there is a $W \subset U$ which is not point–finite such that $f(\{A, B\}) = 1$ whenever $\{A, B\} \in [W]^2$.

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Results from [1] and [3] imply that the following statements are equivalent:

1. For every sequence \((U_n : n = 1, 2, 3, \ldots)\) of \(\omega\)-covers of \(X\), there is a sequence \((U_n : n = 1, 2, 3, \ldots)\) such that for each \(n\) \(U_n \in U_n\), and \(\{U_n : n = 1, 2, 3, \ldots\}\) is an \(\omega\)-cover of \(X\).

2. For each \(n\), \(X^n\) has Rothberger’s property.

3. \(\Omega \rightarrow (\Omega)^2\).

4. For each \(n\) and each \(m\), \(\Omega \rightarrow (\Omega)^m\).

The partition relation in the fourth item means the following:

For every open \(\omega\)-cover \(U\) of \(X\) and for all positive integers \(n\) and \(m\), for each \(f : [U]^n \rightarrow \{1, 2, \ldots, m\}\), there is an \(\omega\)-cover \(V \subset U\) and an \(i \in \{1, 2, \ldots, m\}\) such that \(f(\{A_1, \ldots, A_n\}) = i\) whenever \(\{A_1, \ldots, A_n\} \in [V]^n\).

Since every \(\omega\)-cover is also a large cover, we have the following implications:

\[\Omega \rightarrow (\Omega)^2 \Rightarrow \Omega \rightarrow (\Omega, \Lambda)^n,\]  \(\text{(2)}\)

\[\Omega \rightarrow (\Omega, \Lambda)^n \Rightarrow \Omega \rightarrow (\Omega, \mathcal{O})^n,\]  \(\text{(3)}\)

\[\Omega \rightarrow (\Omega, \Lambda)^n \Rightarrow \Omega \rightarrow (\Lambda)^n,\]  \(\text{(4)}\)

and

\[\Omega \rightarrow (\Lambda)^n \Rightarrow \Omega \rightarrow (\mathcal{O})^2.\]  \(\text{(5)}\)

We investigate the reversibility of these implications.

**Theorem 1** For \(X\) a set of real numbers, the following are equivalent:

1. \(\Omega \rightarrow (\Omega)^2\).

2. \(\Omega \rightarrow (\Omega, \mathcal{O})^2\).

**Proof** : \(1 \Rightarrow 2\) is clear, since every element of \(\Omega\) is an open cover of \(X\).

\(2 \Rightarrow 1\): We shall now use the result from [3] and [1] that \(\Omega \rightarrow (\Omega)^2\) is equivalent to the assertion that for every sequence \((U_n : n = 1, 2, 3, \ldots)\) of
There is a sequence \((U_n : n = 1, 2, 3, \ldots)\) such that for each \(n\) \(U_n \in \mathcal{U}_n\) and \(\{U_n : n = 1, 2, 3, \ldots\}\) is an \(\omega\)-cover of \(X\). Thus, let \((U_n : n = 1, 2, 3, \ldots)\) be a sequence of \(\omega\)-covers of \(X\). We may assume that each is countable; enumerate each \(U_n\) bijectively as \((U^k_n : k = 1, 2, 3, \ldots)\). Now we use a partition from [1]: First define 

\[
\mathcal{V} = \{U^1_k \cap U^1_m : k, m = 1, 2, 3, \ldots\}
\]

and observe that \(\mathcal{V}\) is an \(\omega\)-cover of \(X\). Now define a partition \(f : [\mathcal{V}]^2 \to \{0, 1\}\) by

\[
f(\{U^1_{k_1} \cap U^1_{m_1}, U^1_{k_2} \cap U^1_{m_2}\}) = \begin{cases} 
0 & \text{if } k_1 \neq k_2 \\
1 & \text{otherwise}
\end{cases}
\]

Now if \(\mathcal{W} \subset \mathcal{V}\) is homogeneous of color 1, then there is a \(k\) such that \(\mathcal{W} \subseteq \{U^k_\ell \cap U^k_k : \ell = 1, 2, 3, \ldots\}\), a collection of subsets of the open proper subset \(U^k_k\) of \(X\), and thus \(\mathcal{W}\) is not a cover of \(X\). Thus, by 2, there is an \(\omega\)-cover \(\mathcal{W} \subset \mathcal{V}\) of \(X\) which is homogeneous of color 0 for \(f\). But then \(\mathcal{W}\) is of the form \(\{U^1_{i_1} \cap U^1_{j_1}, U^1_{k_1} \cap U^1_{j_2}, \ldots\}\), where \(i_1 \neq k_1\) whenever \(i_1 \neq j_1\). But then the sequence \((U^1_{i_1}, U^1_{j_1}, \ldots)\) constitutes an \(\omega\)-cover of \(X\). For each \(n\) put \(V^k_n = U^k_{i_n}\) and select for \(n \notin \{k_1, k_2, \ldots\}\) and arbitrary element \(V_n\) of \(U_n\). Then we have found a sequence \((V_1, V_2, \ldots)\) such that for each \(n\) \(V_n \in U_n\) and \(\{V_n : n = 1, 2, 3, \ldots\}\) is an \(\omega\)-cover of \(X\). \(\square\)

Using that same partition one shows:

**Theorem 2** If the collection of open \(\omega\)-covers of \(X\) satisfies the partition relation \(\Omega \to (\mathcal{O})^2\), then \(X\) has Rothberger’s property.

**Proof** : Recall from [3] that a separable metric space \(X\) has Rothberger’s property if, and only if, for every sequence \((\mathcal{U}_n : n < \omega)\) of open \(\omega\)-covers of \(X\), there is a sequence \((U_n : n < \omega)\) such that for each \(n\) \(U_n \in \mathcal{U}_n\), and \(\{U_n : n < \omega\}\) is an open cover of \(X\). Now the partition relation holds for the family of open \(\omega\)-covers of \(X\), and let \((\mathcal{U}_n : n < \omega)\) be a sequence of \(\omega\)-covers of \(X\). Each may be assumed to be countable – let \((U^m_n : m < \omega)\) be a bijective enumeration of \(\mathcal{U}_n\).

Put 

\[
\mathcal{V} = \{U^0_n \cap U^m_k : n, k < \omega\}
\]
and define the partition \( f : [\mathcal{V}]^2 \to \{0, 1\} \) by

\[
f([U_{k_1}^0 \cap U_{m_1}^{k_1}, U_{k_2}^1 \cap U_{m_2}^{k_2}]) = \begin{cases} 
0 & \text{if } k_1 \neq k_2 \\
1 & \text{otherwise}
\end{cases}
\]

Now if \( \mathcal{W} \subset \mathcal{V} \) is homogeneous of color 1, then there is a \( k \) such that
\( \mathcal{W} \subseteq \{U_k^0 \cap U_\ell^k : \ell = 1, 2, 3, \ldots\} \), a collection of subsets of the open proper subset \( U_k^0 \) of \( X \), and thus \( \mathcal{W} \) is not a cover of \( X \). Thus there is an open cover \( \mathcal{W} \subset \mathcal{V} \) of \( X \) which is homogeneous of color 0 for \( f \). But then \( \mathcal{W} \) is of the form \( \{U_k^i \cap U_\ell^j : i = 1, 2, 3, \ldots\} \), where \( k_i \neq k_j \) whenever \( i \neq j \). But then the sequence \( (U_{j_1}^{k_1}, U_{j_2}^{k_2}, \ldots) \) constitutes an open cover of \( X \). For each \( n \) put \( V_n = U_{j_n}^{k_n} \) and select for \( n \not\in \{k_1, k_2, \ldots\} \) and arbitrary element \( V_n \) of \( \mathcal{U}_n \). Then we have found a sequence \( (V_1, V_2, \ldots) \) such that for each \( n \)
\( V_n \in \mathcal{U}_n \) and \( \{V_n : n = 1, 2, 3, \ldots\} \) is an open cover of \( X \). \( \square \)

**Problem 1** Is it true that if \( X \) has Rothberger’s property then its collection of \( \omega \)-covers satisfies the partition relation \( \Omega \to (\mathcal{O})^2 \)?

**Theorem 3 (CH)** There is a set of real numbers which has Rothberger’s property and satisfies for each \( n \) and each \( m \) the partition relation

\[
\Omega \to (\Lambda)^n_m
\]

but does not satisfy the partition relation \( \Omega \to (\Omega)^2 \).

**Proof**: We construct a set of real numbers whose family of \( \omega \)-covers satisfies the former partition relation, but not the latter by adapting a construction from [1].

Let \( ((\mathcal{U}_\alpha, f_\alpha) : \alpha < \omega_1) \) enumerate all pairs \( (\mathcal{U}, f) \) were \( \mathcal{U} \) is a countable collection of open subsets of \( \omega^\omega \) and there are positive integers \( n \) and \( m \) such that \( f \) is a function from \( [\mathcal{U}]^n \) to \( \{1, \ldots, m\} \). Also let \( C \) be a countable dense subset of \( \omega^\omega \), let \( (D_\alpha : \alpha < \omega_1) \) enumerate all dense \( G_\delta \) subsets of \( \omega^\omega \), and let \( (h_\alpha : \alpha < \omega_1) \) enumerate \( \omega^\omega \).

Construct \( X = \{g_\alpha : \alpha < \omega_1\} \cup \{\ell_\alpha : \alpha < \omega_1\} \cup C \) recursively as follows:

For \( \alpha = 0 \):

**Case 1**: \( \mathcal{U}_0 \) is an \( \omega \)-cover of \( C \); Then choose, by the countability of \( C \), an infinite subset \( \mathcal{W}_0 \) of \( \mathcal{U}_0 \) such that every element of \( C \) is in all but finitely
many elements of \( W_0 \), and then apply Ramsey’s theorem for \( f_0 \) to \( W_0 \) to find an infinite subset \( V_0 \) of \( W_0 \), which is homogeneous for \( f_0 \).

Let \( (A^0_k : k < \omega) \) enumerate the finite subsets of \( C \), and for each \( k \), define:

\[
O^0_k = \bigcup \{ V \in V_0 : A^0_k \subset V \}.
\]

Then each \( O^0_k \) is a dense open subset of \( \omega^\omega \), and so

\[
D_0 \cap (\cap_{k<\omega} O^0_k)\]

is a dense \( G_\delta \) set. Choose elements \( g_0 \) and \( \ell_0 \) of it such that for each \( n \) we have \( \max\{g_0(n), \ell_0(n)\} > h_0(n) \). This then specifies a triple \( (V_0, g_0, \ell_0) \) such that:

1. \( V_0 \) is infinite,
2. each element of \( C \) is in all but finitely many elements of \( V_0 \),
3. \( V_0 \) is a large cover of \( C \cup \{g_\gamma : \gamma < \alpha\} \cup \{\ell_\gamma : \gamma < \alpha\} \), and
4. \( V_0 \) is homogeneous for \( f_0 \).

**Case 2:** \( U_0 \) is not an \( \omega \)-cover of \( C \): Then we set \( V_0 = \emptyset \), put \( O^0_k = \omega^\omega \) for each \( k \), and we choose \( g_0 \) and \( \ell_0 \) as above.

\( \alpha > 0 \): Now assume that for each \( \beta < \alpha \) we have selected a triple \( (V_\beta, g_\beta, \ell_\beta) \) such that:

1. If \( V_\beta \) is nonempty, then it is an infinite subset of \( U_\beta \),
2. If \( V_\beta \) is nonempty, then each element of the countable set \( C \cup \{g_\gamma : \gamma < \beta\} \cup \{\ell_\gamma : \gamma < \beta\} \) is in all but finitely many of the elements of \( V_\beta \),
3. If \( V_\beta \) is nonempty, then it is a large cover for \( C \cup \{g_\gamma : \gamma < \alpha\} \cup \{\ell_\gamma : \gamma < \alpha\} \),
4. If \( V_\beta \) is nonempty, then it is homogeneous for \( f_\beta \),
5. for each \( n \), \( \max\{g_\beta(n), \ell_\beta(n)\} > h_\beta(n) \), and
6. \( \{g_\beta, \ell_\beta\} \subset \cap_{\gamma \leq \beta} D_\gamma \).
Case 1: \( \mathcal{U}_\alpha \) is an \( \omega \)-cover of
\[
C \cup \{g_\gamma : \gamma < \alpha\} \cup \{\ell_\gamma : \gamma < \alpha\}.
\]
Let \( C_\alpha \) denote this set. Then select an infinite subset \( \mathcal{W}_\alpha \) of \( \mathcal{U}_\alpha \) such that each element of \( C_\alpha \) is in all but finitely many elements of \( \mathcal{W}_\alpha \); then apply Ramsey's theorem to the pair \( (\mathcal{W}_\alpha, f_\alpha) \) to find an infinite subset \( \mathcal{V}_\alpha \) of \( \mathcal{W}_\alpha \) which is homogeneous for \( f_\alpha \).

Next, let \( (A_\alpha^k : k < \omega) \) enumerate the finite subsets of \( C_\alpha \), and define, for each \( k \),
\[
O_\alpha^k = \bigcup \{V \in \mathcal{V}_\alpha : A_\alpha^k \subset V\}.
\]
Then each \( O_\alpha^k \) is a dense open set, whence
\[
(\cap_{\gamma \leq \alpha} D_\gamma) \cap (\cap_{k < \omega} O_\alpha^k)
\]
is a dense \( G_\delta \)-set. Next choose \( g_\alpha \) and \( \ell_\alpha \) from this intersection such that for each \( n \) we have
\[
\max\{g_\alpha(n), \ell_\alpha(n)\} > h_\alpha(n).
\]
Then \( (\mathcal{V}_\gamma, g_\gamma, \ell_\gamma), \gamma \leq \alpha \) also satisfy the six induction hypotheses.

Case 2: \( \mathcal{U}_\alpha \) is not an \( \omega \)-cover of
\[
C \cup \{g_\gamma : \gamma < \alpha\} \cup \{\ell_\gamma : \gamma < \alpha\}.
\]
Then we put \( \mathcal{V}_\alpha = \emptyset \), for each \( k \) we put \( O_\alpha^k = \omega \), and we choose \( g_\alpha \) and \( \ell_\alpha \) as before. Once again, the six induction hypotheses are satisfied by the triples \( (\mathcal{V}_\gamma, g_\gamma, \ell_\gamma), \gamma \leq \alpha \).

This describes the recursive construction of \( X \). The sixth item in the induction hypotheses guarantees that \( X \) is a Lusin set and as such has Rothberger's property. We must see that the claimed facts about the partition relations are also true.

Now let \( \mathcal{U} \) be a (countable) open \( \omega \)-cover of \( X \) and let
\[
f : [\mathcal{U}]^n \to \{1, 2, \ldots, m\}
\]
be a coloring. Fix an \( \alpha \) such that \( (\mathcal{U}, f) = (\mathcal{U}_\alpha, f_\alpha) \).

Since \( \mathcal{U}_\alpha \) is an \( \omega \)-cover of \( (C_\alpha = \{g_\gamma : \gamma < \alpha\} \cup \{\ell_\gamma : \gamma < \alpha\} \cup C) \) at stage \( \alpha \) we have selected \( \mathcal{V}_\alpha \subset \mathcal{U}_\alpha \), an \( \omega \)-cover of \( C_\alpha \), which is homogeneous for \( f_\alpha \) (items 2 and 4 of the induction hypotheses), and is a large cover for \( X \) (item 3 of the induction hypotheses). This establishes that the partition relation
\[
\Omega \to (A)^n_m
\]
holds for all positive integers $n$ and $m$. To see that the partition relation
\[ \Omega \rightarrow (\Omega)_2^2 \]
does not hold, we show that there is a sequence $(U_n : n = 1, 2, 3, \ldots)$ of $\omega$-covers of $X$ such that for each sequence $(U_n : n = 1, 2, 3, \ldots)$ where for all $n$, $U_n \in U_n$, \{U_n : n = 1, 2, 3, \ldots\} is not an $\omega$-cover of $X$. To this end define for each $n$ and $k \in \omega$ 

\[ U^n_k = \{ f \in \omega : f(n) \leq k \} \]

and then put $U_n = \{U^n_k : k = 0, 1, 2, \ldots\}$. Then each $U_n$ is an $\omega$-cover of $X$. Consider any sequence $(U^n_k : n = 1, 2, 3, \ldots)$: Then for some $\beta < \omega_1$ we have that for each $n$, $h_\beta(n) = k_n$. Then by item 5 of the induction hypothesis we see that there is no $n$ such that the elements $g_\beta$ and $\ell_\beta$ of $X$ are both in $U^n_k$. This implies that \{U^n_k : n = 1, 2, 3, \ldots\} is not an $\omega$-cover of $X$. □

References


Department of Mathematics
Boise State University
Boise, Idaho 83725
e-mail: marion@math.idbsu.edu